PHASE SPACE METHODS FOR THE ANALYSIS AND DESIGN OF DISCONTINUOUS SYSTEMS

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ABSTRACT

The phase space theory and its application to the analysis and design of discontinuous systems has been discussed for both linear and saturated systems with step and ramp inputs.

The general approach is to analyze the futures of the trajectories in error space, and design various control computers to change the character of the system at proper instants to make the system have fast and nearly dead beat response.

For the purpose of applying the theory to physical systems, the idea of using low order signals to control high order systems has been investigated. And the idea of combining a continuous system with a relay servo system to form three mode operation has been introduced.

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LIST OF SYMBOLS

ρ	Damping factor
α	Attenuation factor
ω_n	Natural frequency
x, E, ε	Error
$\dot{x}, \dot{E}, \dot{\theta}_c, \dot{\theta}_o$	Derivatives with respect to time
θ_R, θ_i	Input quantity
$\overline{\theta}_R, \overline{\theta}_i$	Transformed input quantity
N, n	Integer
$A, a; B, b$	Coefficients
r	Root of characteristic equation
Υ	Pole of open loop function
S	Laplace operator
D	Differentiation operator
P_s	Location of the state point in phase space
$G(s), F_o(s)$	Open loop transfer function in the forward path
$F_c(s)$	Closed loop transfer function
$D_{F_c}(s)$	Denominator of the closed loop transfer function
$H(s)$	Open loop transfer function in a feedback path
K	Forward gain of open loop system
K_T, H, h	Coefficient of tachometer feedback
K_a	Coefficient of acceleration signal feedback
T, t	Time
P_1, P_2	Poles of open loop transfer function in the forward path
$\times$	Pole on root locus plot
$\odot$	Zero on root locus plot
$\blacksquare$	Root on root locus plot
V_s	Output of saturated amplifier
u, v, w	Coordinate axes of transformed error space

CHAPTER I - GENERAL DISCUSSION ABOUT DISCONTINUOUS SYSTEMS.

INTRODUCTION

In the investigations of control systems there are many individual works which can be included in the general classification of discontinuous systems.

In the literature usually the work that has been done is to analyze a specific system with a specific input. This is because the situations or the problems encountered are different for each type of discontinuous system. In this first chapter a general discussion for a generalized control system will be given. The main purpose is to look at the discontinuous system as a whole problem with a general basic theory and then analyze all the possible cases to which this discontinuous theory can be applied. Since the main theory of discontinuous systems is for the purpose of designing a properly controlled switching action, therefore, we may regard the discontinuous system as a special case of the so called optimum relay control system or vice-versa. But the emphasis that needs to be put on the theory of discontinuous systems is that the switching operation is to change the character of the system at a proper instant rather than only change the polarity of the driving torque. In some cases the discontinuous system may have no switch at all, but the change of character of the system is according to the basic theory of discontinuous systems. We will analyze such kind of systems also.

SECTION I - BASIC THEORY OF DISCONTINUOUS SYSTEMS.

First consider how many possible cases there are to which one can apply discontinuous theory. Fig. 1-1 is a general block diagram for discontinuous control systems. The whole control system is controlled by the control computer, which may be a digital or analog computer, an amplifier with relay, or a simple relay circuit. The control computer may receive signals from all the points in the system where signals can be measured or generated. The input, output and the error signals are usually the important ones. Also it has the ability to use its input signals and provide other useful signals that can be derived from its input signals and generate its output signals to control the parameters of the plant, the controller and amplifier, or the feedback signals.

The input signal analyser may be regarded as a part of the control computer. It analyzes the input signal to provide the required signals for the control computer. It may be a differentiator, integrator, a sampler, a simple potentiometer, or a particular function generator. Its output may be the same as the input signal, or the input signal plus a feed forward path for compensating purposes.

In the block of controller and amplifier, there may be an integrator, an amplifier with relay, with R-C network; or a low power amplifier circuit which provides a compensated signal for the high power amplifier to drive the plant. The changeable parameters that can be controlled by the control computer are the gain of the amplifier, and the compensating circuit in the controller.

The output of the amplifier may be in various forms according to what kind of system or plant is to be controlled.

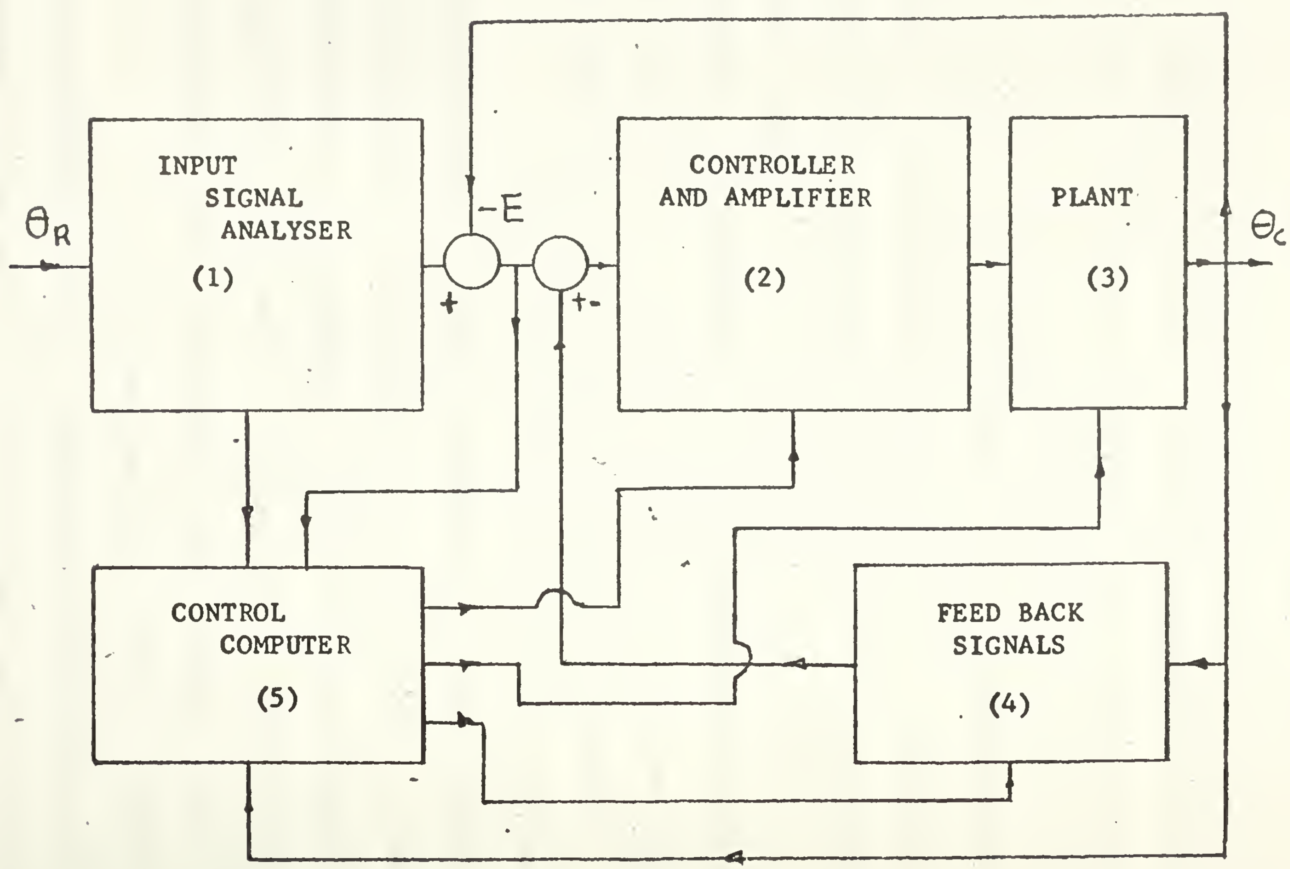


Fig. 1-1 General Block Diagram for Discontinuous System

The plant or load is the object to be controlled by all the other devices. Its output or operating character is designed to meet the specifications of the system. For example in electrical control systems the plant may consist of a motor driving an inertia and friction load; its output shaft is designed to follow the command of the input signal. The parameters that can be controlled by the control computer are the gain, the damping and/or the inertia. The specifications are settling time, overshoot, lagging error for ramp input etc.

The feedback block consists of various signals that can be measured or generated by potentiometer, tachometer, differentiator or integrator by using the output signal. The polarity, instant and period for its output to be applied to the controller is controlled by the control computer. Also the gains of the feed back signals and even their non-linearities can also be controlled by the control computer.

The nonlinearities may exist in any part of the control system. The control computer may only control or use a part of them or neglect all of them depending upon the requirement of the application.

If some feed forward and feed backward paths around each block are added, then the system becomes more complex. Here consider that all of these paths are included in each block already. If some specifications or limitations are put on the blocks then a specific control system occurs. For example, by eliminating block (1), (4) & (5), let the controller take the job of the control computer, then this is a continuous servo system without feed back compensation. If in block (2) there is only an amplifier and a relay, then this is a simple relay servo system. If block (3) is a 2nd order plant and block (4) is continuous tachometer feedback, then this is a 2nd order relay servo system with tachometer feedback to shift

the switching line of the relay operation. The parameters of the plant can also be controlled by the controller or by the plant itself, using dynamic braking or stored energy braking. These had been discussed in detail by McDonald and Thaler (1957-1959).

The poles of the plant may be real or complex, the controller may produce derivative signals by differentiating the error signal. By using these two blocks (2) & (3), various control methods have been developed. The basic theory along this branch of development is to design the controller to make the switch operate along a curve in the phase plane or a surface in phase space, that is the so called optimum response switching curve or hypersurface. A very general analysis has been given by Fuller in his PhD thesis (1959). The phase space analysis theory applied to such systems has also been discussed in his papers (1959, 1960). A plant with complex poles has been discussed by Lotz (1960).

For continuous systems (Block (2) consists of controller and amplifier) by using block (2) & (3), the so called cascade compensated system has been developed. Various papers discussing methods of finding the proper values of parameters in compensator circuits have been published. The theory of lead-lag network compensation seems predominant in this branch of development. Papers based on the root locus approach have been given by Ross, Warren, Thaler (1960) and Mariotti (1960).

If the feed back path from output to controller is taken off, then this is an open loop system, it is useful in some application, but the usual servo system is based on the assumption that the output follows the input and the final error is to be made as small as possible. So this kind of system will not be discussed in the later chapters.

Another approach to improve the response character of a control

system is the feedback compensation method. The simple case of feeding back the derivative signal of output is commonly used for compensating 2nd or some 3rd order systems. The general block diagram consists of blocks (2), (3), and (4). The basic theory is to change the coefficients of the characteristic equation to meet the specifications. The root locus method is a powerful tool for analysis and design. By the recently published papers the relation between transient character and the location of the dominant poles are clearly defined, also the phase margin and gain margin in frequency response can also be derived from the root locus method, If all the derivative signals of the output can be obtained, then such a system for linear operation can be designed very well to meet the specifications. But unfortunately with high order systems the derivative signals are not easily obtainable. On the other hand, in high gain systems the main amplifier tends to be saturated even in 2nd order systems. The theory that regards the saturation of main amplifier as just a relay operation has been used by many authors.

Before the block diagram presentation is made complex, by adding the control computer and input signal analyser, it may be helpful to look into the discontinuous damping theory first.

The characters of discontinuous damping are possessed by many moving creatures in nature. The flying of the bird, the swimming of the fish, the running of the animal and, etc., because this is the best way to use limited physical strength (power) to accomplish traveling in a minimum amount of time. For example in a race of sport cars, if the rule decided that the first prize will be granted to the person who can control the car to use a minimum of time to stop the car at a pre-selected spot with no over-shoot and no under-shoot, then the discontinuous damping method (braking) will play an important part. Maybe the manufacturing company will design the car not only to have a heavy

duty braking system, but also, may redesign the engine to have backward torque in the braking period. Other examples, a bird flying with high speed, before stopping on an object, first spans the wings and then stops. Why not just reduce the propulsion and let the speed go to zero and then stop? The answer is quite clear that the bird has a discontinuous system to use and the discontinuous system gives fast flying ability and best stability. Other authors use the running of a train as an example for discontinuous sytems. In such an example the fact of economic consideration can also prove the discontinuous sytem is the better one.

Returning to automatic control terminology, some comparisons with other systems are helpful.

The so called optimization switching theory is to instrument the controller of the system so that it will follow an optimum trajectory in space, the hypersurface used for switching is decided by the input and error signals and their derivatives, the trajectory stays always at either side of the hypersurface, i.e., full voltage is applied to the system at all times. But the discontinuous system is based on the theory that it is best to use maximum drive at the beginning part of the trajectory, and to brake the system at a proper point on the trajectory. In other words, the character of the system is entirely different when the state point is at different sides of the hypersurface.* The die out part of the trajectory should be made as near the $\ddot{E}$ vs. $\dot{E}$ plane as possible. From the projection view of the E vs $\dot{E}$ plane, the general trajectory for a step input will be as in Fig. 1-2. The system may be linear or nonlinear, as long as the die-out trajectory can be made to stay or nearly stay in a state corresponding to overdamped systems, then the control computer can be simplified and the response can be

*See Fuller (1960)

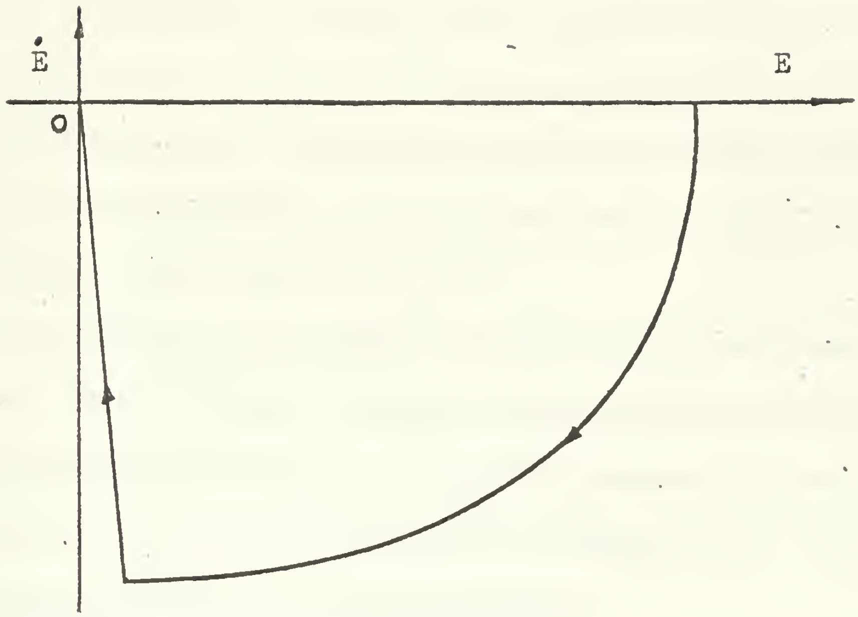


Fig.1-2 Typical Response Curve of Discontinuous System Due To Step Input

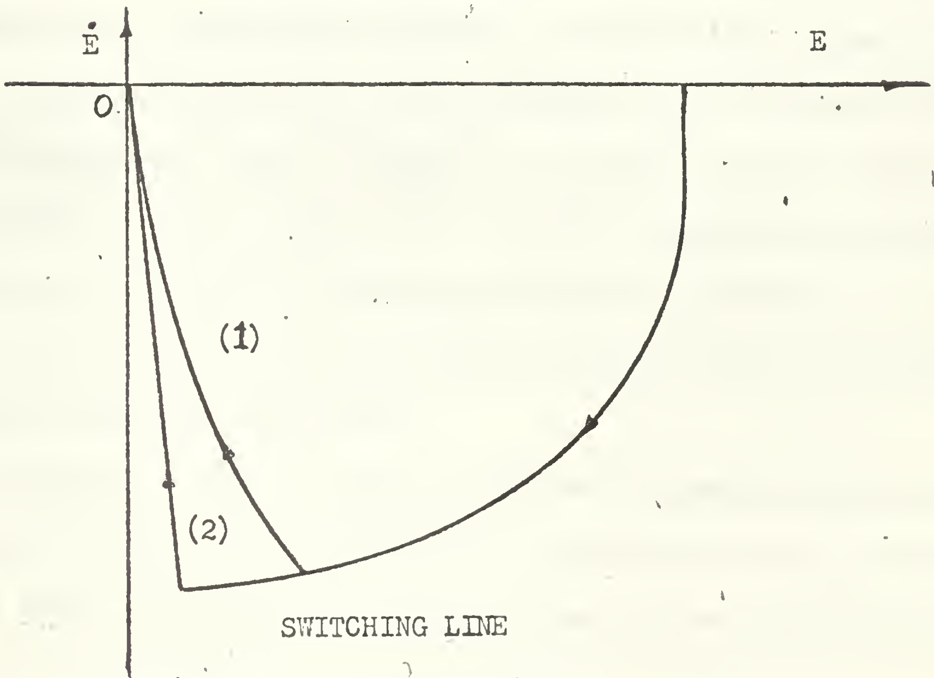


Fig.1-3 Trajectory Comparison
(1)Ordinary Optimization
(2)Discontinuous System

improved. For example, in the second order relay servo system or a high gain system with saturation in the main amplifier, the difference in trajectory between an optimized switching system and the discontinuous system is indicated in Fig. 1-3. In case the die-out part of the trajectory can not be made a straight line, then the use of linear switching methods to approximate the die-out trajectory is still applicable. Typical trajectories for step inputs are plotted in Fig. 1-4.

Since the switching action changes the character of the system at the same time, and a heavily damped (braking) system is used in the die-out part of the trajectory, then the die-out part of the trajectory is very near the $\dot{E}$ axis. It may be a straight line or near a straight line. Then the problem of controlling the switch can be simplified.

The main point of the discontinuous system is to gain the desired response character by changing the character of the original system to make the linear switching method* at the die-out part of the trajectory possible. In other words the main point of design is to create a better acceleration and deceleration trajectory rather than follow the original so called optimum trajectory. The optimized switching theory can be applied to this continuous system also, but for many applications the linear switching method is good enough to be accepted.

In a comparison with the continuous system, the phase plane method may be used to illustrate the difference. A continuous system, is usually designed to have some overshoot in the later part of the trajectory, because

* Linear switching method - The operation of the relay is based on a linear differential equation, in 2nd order system it is a line; in 3rd order system it is a plane.

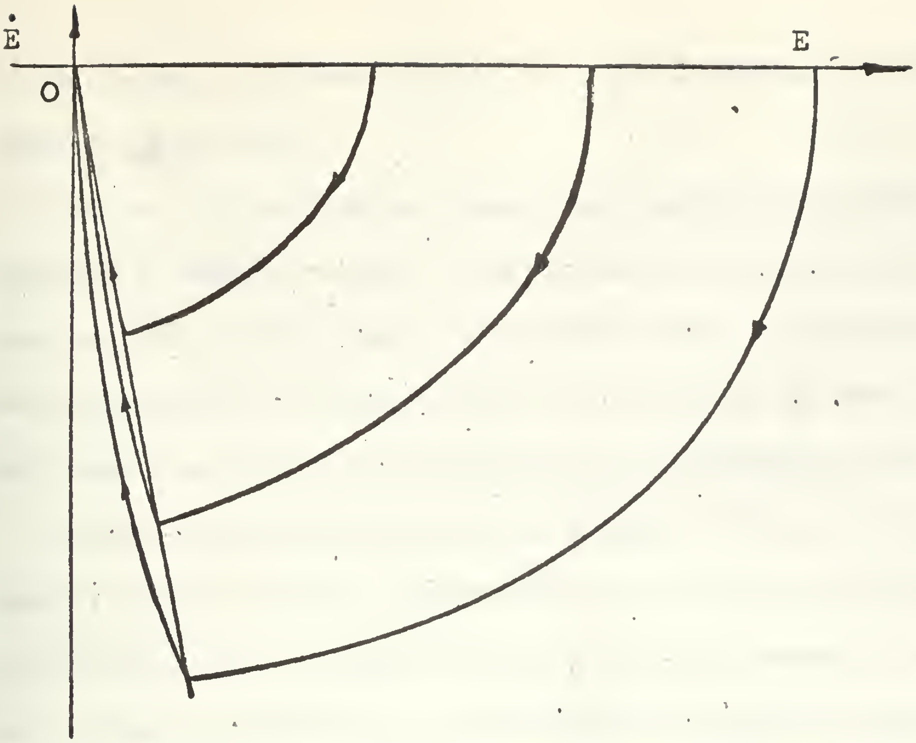


Fig. 1-4 Approximation for Step Inputs by Linear Switching Method

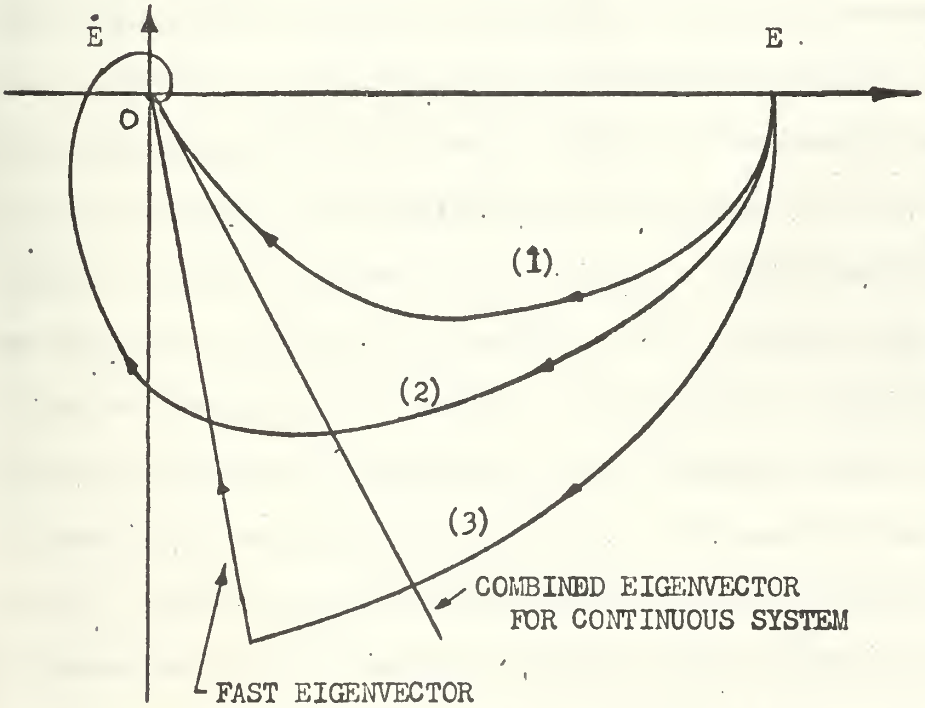


Fig. 1- 5 Comparison Between Continuous and Discontinuous Systems.(I)&(2)Continuous System,(3)Discontinuous System.

if the system is operated in the critical or overdamped condition the response is quite slow.

But in a discontinuous system the acceleration trajectory can be made unstable or nearly unstable. The die-out trajectory can be made a straight line near the vertical axis in the phase plane (fast eigenvector). By throwing the switch at a proper point, the trajectory is like that in Fig. 1-2. The comparison with a continuous system is indicated in Fig. 1-5.

Returning to the general block diagram in Fig. 1-1, consider what the control computer can do. Assume that there is no input signal analyser and there is only one output from the control computer to control the feedback signals. Then this is a discontinuous feedback system. If the controller and amplifier are continuous then the feedback signals compensate the system according to the output of the control computer. The feedback may be positive or negative, discrete, or with non-linearities. But the main purpose is to create a fast acceleration trajectory and a heavily damped die-out trajectory. The paper written by Ostrovskii (1958) gave a clear analysis, though it only discussed the step input and the form of the characteristic equation is also limited. Meiksin (1958) analyzed such systems by using positive and negative feedback. They both have made contributions along this branch of development. The limitation of this discontinuous feedback system are the availability of feedback signals and the saturation in both the forward and backward paths. For example in an electric servo system, if the derivative signal is produced by a tachometer and a simple differentiator, then the noise, and non-linear problems will become severe even in the 2nd order derivative signal. On the other hand, after the main amplifier is saturated, the feedback signals lose their control ability, the system will have a relay servo character, and the trajectory will be decided only by the maximum output of the amplifier and the character of the plant.

In this case the feedback signals can only do the switching job. Since the switching operation can be controlled by the feedback signals themselves, the control computer can even be eliminated. But in order to make the system have a good response for input signals other than a step input, the control computer will still play an important part. This will be discussed in later chapters.

If an output is added to the control computer which controls the controller and amplifier, then it is a complex discontinuously compensated system. The term "complex" is used because it is neither a simple feedback compensated system nor a simple cascade compensated system. To simplify this system take off the feedback block, then it becomes a discontinuous cascade compensated system. In the controller part, the gain of the amplifier, the error signal and its derivatives or integrated signals can be discontinuously controlled to provide an under or overdamped character.

A switching action which inserts (or removes) the cascade compensators can also do the same job. In some cases even control of an on and off switch can make the system have dead beat response for some particular input, such as the posicast method of control given by Smith (1957) and So & Thaler (1960).

The combination system with discontinuous feedback and cascade compensation may have various performances due to the different control methods used in each of the blocks(2)&(4). The initial condition and nonlinearity effect will become more complex. But this is a possible way to approach the problem of improving the character of the control system. It needs some investigation to show how much improvement can be obtained.

If only the output of the control computer is used to control the plant, then this is another branch of discontinuous control system. The changeable parameters in block(3)include the gain of the motor, the inertia,

the friction or a special designed damping device such as electrical damper (discussed in later chapters), or the change of the polarity of the motor action such as three mode operation (discussed in Chapter VI). The theory of dynamic braking and stored energy braking may also be included in the general theory of discontinuous damping along this branch.

If adding block (4) to the above discontinuous system, the compensation character will be improved, a higher order plant may be controlled better than before. Again block (2) may be a linear amplifier, a saturated amplifier, or a relay circuit controlled by the output of the amplifier. The feedback signal may be used to compensate the system or only for switching purposes. A relay servo with continuous tachometer feedback and with discontinuous friction damper may be considered as a good example. The tachometer feedback controls the switching line, and the friction damper makes the die-out trajectory nearly a straight line which stays in the dead zone of the relay or nearly in the dead zone. By using discontinuous feedback, the changeable parameters are more than that in the above system. The discontinuous operations in block (3) & (4) may occur at the same instant or may be controlled separately. The controller and amplifier may be linear or non-linear. Among the various combinations take as an example a discontinuous feedback linear system (or piecewise linear) with discontinuously controlled plant parameter (gain or friction). The contribution of each discontinuously controlled block is just as mentioned before.

All of the four blocks (2), (3), (4) and (5) may be used. The rules for explaining the characters of each combination is just as before, first by assuming some possible character for each block and then analyzing the combined system to see if there is improvement.

If signals can be taken out from the blocks (2), (3) & (4), that are controlled

by the output of control computers, then these outputs from block (2), (3) & (4) can be used to feed into the control computer. The so called self-adaptive system has had long range progress along this line. The usually adopted method is to assume a non-linear parameter in a block and all the other parts of the system are linear. The general idea is to have a control computer which is affected by this non-linear parameter, and the output of the control computer is used to change another parameter to compensate the non-linearity effect. In discontinuous systems the same theory can be applied. The relation between the non-linear parameter and the output of the control computer may be designed to be different in the acceleration part and deceleration part of the trajectory. In other words, the non-linear parameter may also possibly be used to improve the character of the system.

Up to now the discussion has been general, concerning the possible ways to use discontinuous theory to improve the response character of systems. Also some comparisons have been made with the simple step input case. But the general switching theory and the response character of the system with other kind of input signals have not been discussed yet. In order to take this step it is necessary to have an understanding of the phase space analysis and design theory.

SECTION II - PHASE SPACE ANALYSIS AND DESIGN

The basic concept of phase space is "choosing a set of time variables of the system as the coordinate axes, the futures of a particular set of time variable can be completely determined by their coordinate values (initial conditions) and the character of the forcing function. (For detail see Fuller's paper, 1960). But for discontinuous systems one needs a description that is the futures of a particular set of time variable not only decided by the initial conditions and the character of the forcing function but also decided by the different characters of the system, because the switching action is combined with the changing of the system character to give optimum or fast response.

A Review of the Phase Plane Method:

Before studying the phase space theory it is helpful to review the phase plane theory. The commonly used phase plane uses error (E) and its derivative with respect to time ($\dot{E}$) as the axes in the Cartesian coordinates. The form of the trajectories of under-and overdamped cases are sketched in Fig. 1-6a, b.

By using discontinuous damping, the trajectory can be made as in Fig. 1-6c (this part will be discussed in detail in Chapter II). Choosing a switching line which has a smaller slope (less negative) than the fast eigenvector, then the die-out trajectory will follow this switching line but may have oscillation (chattering) as in Fig. 1-6d. If the switching line has a larger slope than the fast eigenvector, then there will be over shoot as in Fig. 1-6d.

In some systems the switching line with smaller slope than the fast eigenvector is preferable, because this switching line causes no under shoot or over shoot, the switching point is not critical, the trajectory after

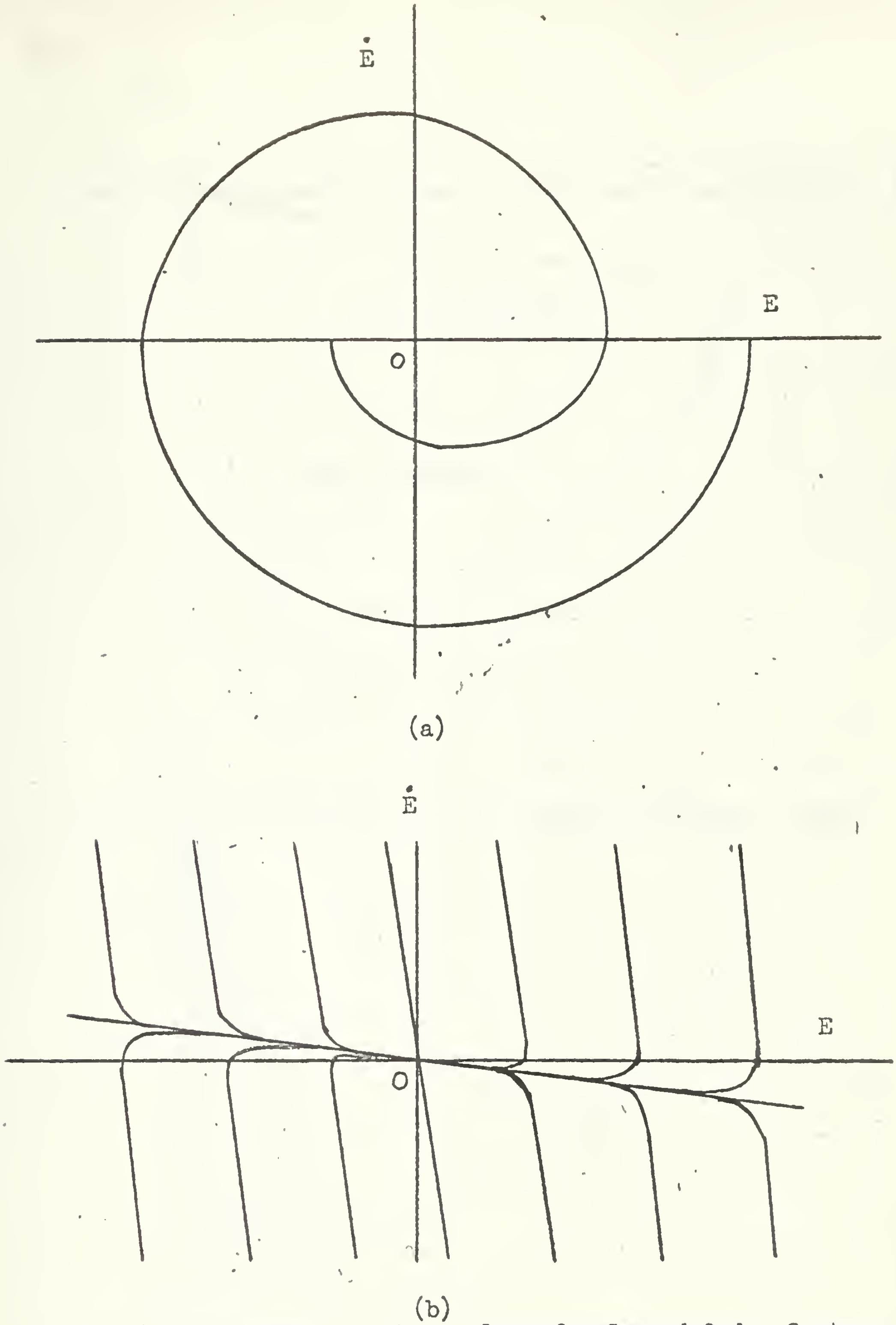


Fig.1-6 Trajectories in Phase Plane for Second Order System.

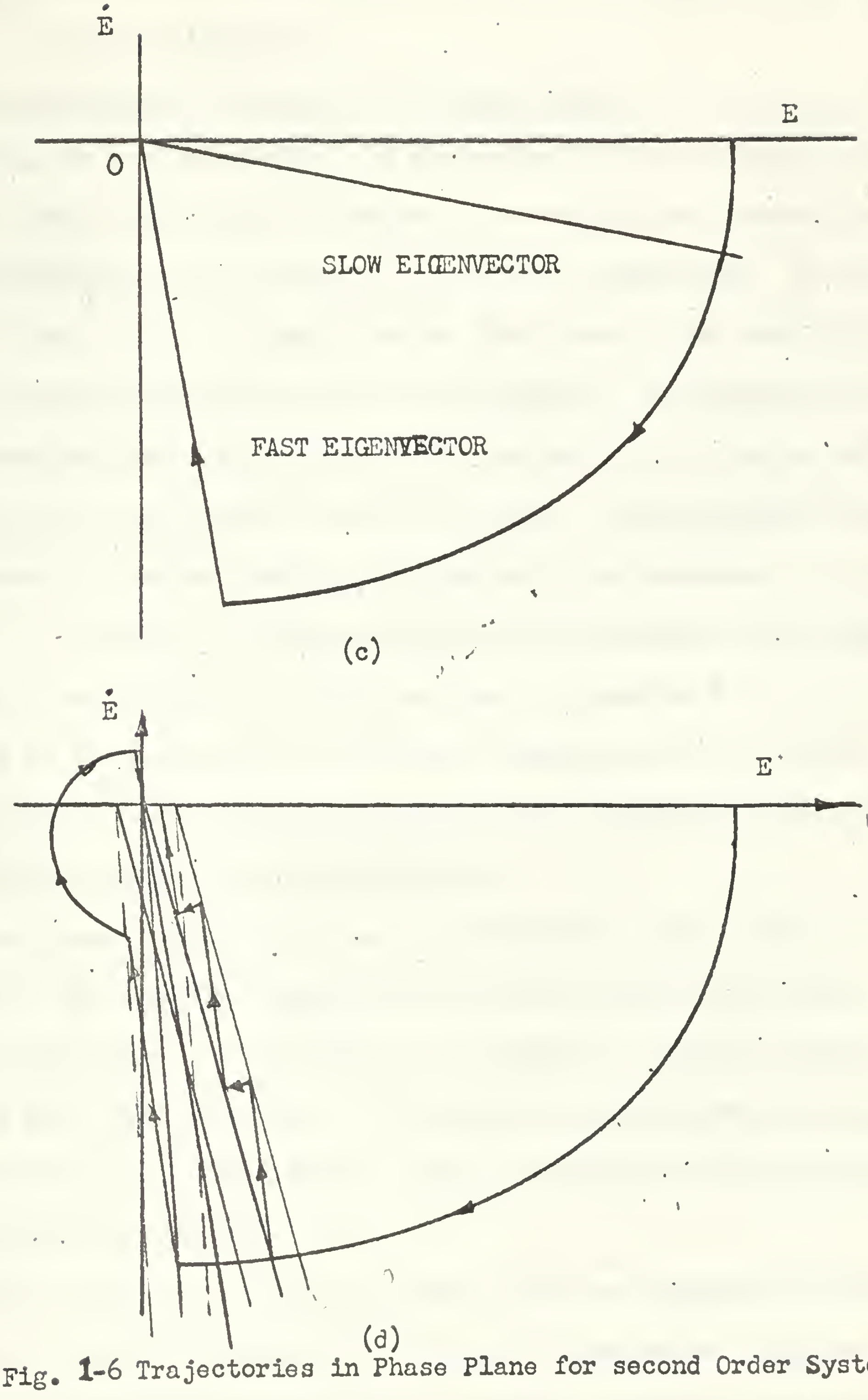


Fig. 1-6 Trajectories in Phase Plane for second Order System

entering into the switching line will stay on the switching line. The chattering can be eliminated by choosing the slope of the switching line near that of the fast eigenvector.

<u>The Trajectory Due to Step Input in Error Space</u>:

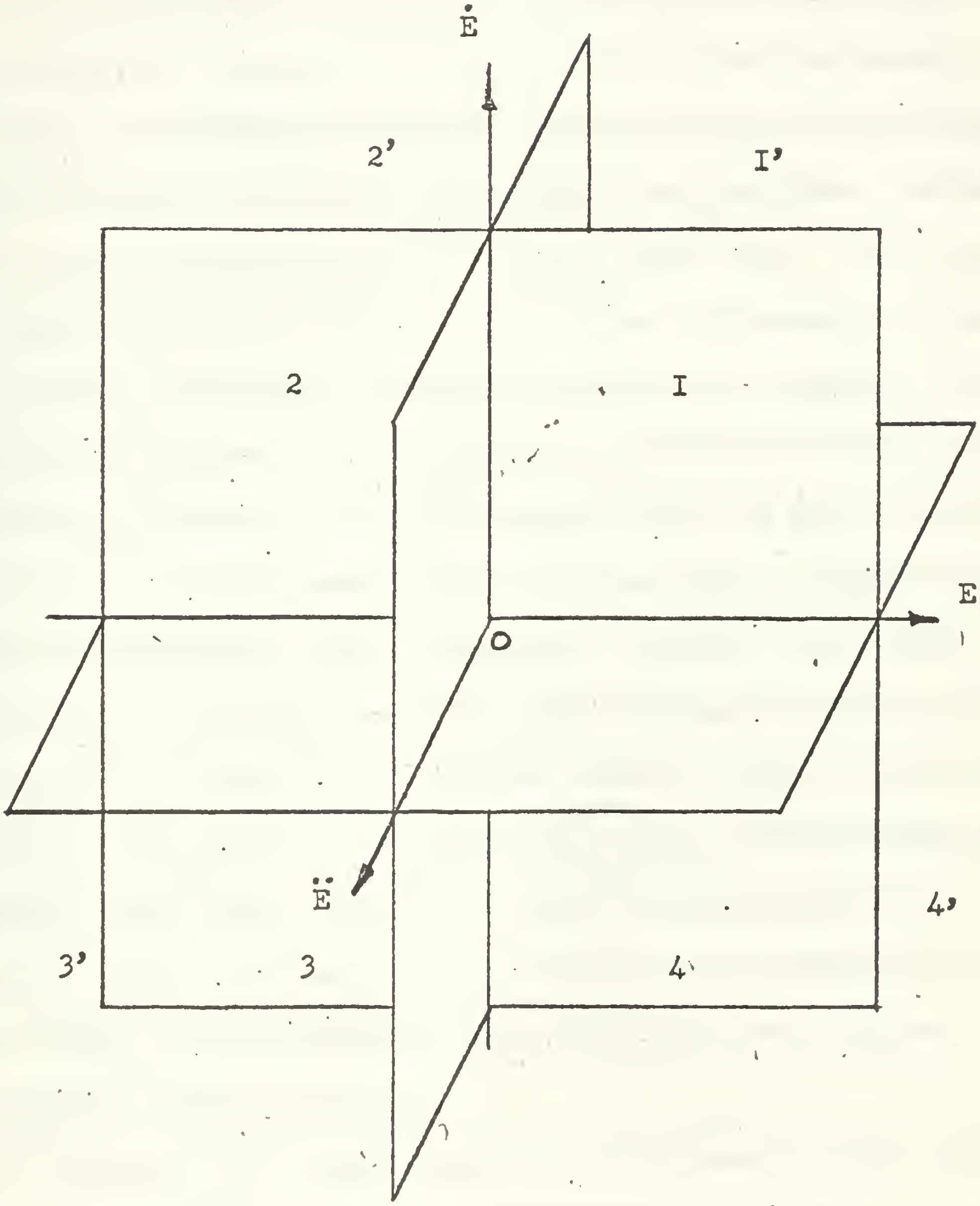
By using the analog computer the projection of the trajectory in various coordinate planes can be easily obtained. For third order systems a model of three dimensional error space (E, $\dot{E}$, $\ddot{E}$) can be constructed. For any value of positive step input, the trajectory at first goes in the negative direction of $\ddot{E}$ axis and then curves back to the positive $\ddot{E}$ direction, as the error is reducing (for stable systems) the trajectory is in spiral shape, the final point is the origin of the error space. The underdamped trajectory travels in various quadrants according to the sequence, 4', 4, 3, 2, 2', 1', 4' .....in Fig. 1-7. The trajectory corresponding to the stable limit case is the one with its final part only in quadrants 4', 3, 2, 1', 4', ...that is an ellipse with one diameter combined with the $\dot{E}$ axis. All the trajectories starting from any set of initial conditions in this phase space will tend to reach this final ellipse.

The overdamped trajectory due to a positive step input stays in 4' and 4 quadrants. The more the damping the nearer the trajectory to the E axis. The phase trajectories corresponding to intermediate cases are spirals with their final part kept in a plane. The slopes of these planes (the intersection line with the E vs $\dot{E}$ plane) tend to be reduced (less negative) when the damping is increased.

By using a switch to change the system from the underdamped case to the overdamped case at various points along the underdamped trajectory and with various magnitude of step inputs or initial conditions, one finds that there are three hyper-planes* in this third order error space, in which

* See Ostrovskii (1958) or Fuller (1960d).

Fig. 1-7 Coordinate Planes of Third Order Phase Space.



the over damped trajectories starting at any point on these planes will stay in that plane and go to the origin of the error space. Any trajectory which starts at a point which is not on these three planes will tend to be tangent to that one of these hyper-planes which (near that starting point) has a less negative slope to the E axis. If the damping is very heavy, especially for an overdamped system with one small root and two large roots in the characteristic equation, then the last part of the overdamped trajectory will tend to be tangent to the slow eigenvector which is approximately a straight line near the E axis corresponding to the continuous overdamped trajectory, and the state point will move very slowly there. The projection of the trajectory on to the E vs $\dot{E}$ or $\ddot{E}$ vs $\dot{E}$ plane will appear as an undershoot or overshoot. Therefore, if one makes the control computer switch from the underdamped trajectory to the overdamped trajectory just at the point where the underdamped trajectory hits the hyper-plane this will give fast response; this is just like the requirement in the 2nd order case, to switch from the underdamped trajectory on a fast eigenvector. In this 3rd order error space one can also use a switching plane which passes through the origin but a little above the fast hyper-plane (has less negative slope), then the trajectory on both sides of this plane will tend to come into this hyper-plane. This converged case is just like the 2nd order case mentioned before. Then by making the system underdamped at one side (upper) and overdamped at the other side (lower) the discontinuous system will give fast response to step input or various initial conditions.

This statement may be extrapolated to the nth order system by saying this: By using discontinuous methods to produce an under and overdamped trajectory of the system for step input, fast response can be obtained by using hyper-surface* switching in the error space.

*Hyper-surface (or hyper-space, hyper-plane, eigenspace, eigenplane, it is a plane in 3rd order space, a line in phase plane where the overdamped trajectories stay in it, if starts in it.)

The mathematical proof of this statement has been given by Ostrovskii (1958). Also the switching operations can be reduced to minimum if the die-out part of the trajectory can be made to stay in the hyper-surface (See Chapter II).

<u>The Effect of Driving Function</u>:

The differential equation of an type one nth order system with driving function is:

$$A_0 E + A_1 \dot{E} + A_2 \ddot{E} + \cdots\cdots A_{n-1} \overset{n-1}{E} + A_n \overset{n}{E} = f(\dot{\theta}_R, \ddot{\theta}_R \cdots) \qquad (1\text{-}1)$$

Since the type one system can not follow the acceleration signal input or higher order derivative signals, the differential equation may be simplified as:

$$A_0 E + A_1 E + A_2 \ddot{E} + \cdots A_{n-1} \overset{n-1}{E} + A_n \overset{n}{E} = A_1 \omega_i \qquad (1\text{-}2)$$

where ω_i is the magnitude of the ramp input.

By comparing this equation with the well known characteristic equation of the linear system with a unit step input, the effect of the ramp input is to translate the original error space to the positive or negative error direction by an amount of $\frac{A_1 \omega_i}{A_0}$, (the lagging error). When feedback of the tachometer output is used to damp the system then the lagging error will be $(K_T + \frac{A_1}{A_0})\omega_i$. For a heavily damped case this lagging error may be very large. Then another way must be found to design the control computer to reduce this large lagging error. Detailed discusstions about this problem will be given in Chapter V. One thing to consider is the character of the input signal which can be used to control the switching hypersurface. By feeding the input signal and its derivatives into the control computer, it is possible to design a control computer which gives proper switching action for various input signals, therefore, an input signal analyser is needed. The input signal analyser may be considered as included in the control computer.

The discussion in the above paragraphs about the effect of driving

function seems to lose its generality. The switching operation that is based on a generalized mathematical approach has been given by Fuller (1960). This study is not only concerned with the switching problem but also with the damping used to change the character of the system.

Among the ways to solve the problems in control systems there are two extreme cases. The one is to assume the computer can do anything. Once the mathematical solution is obtained then the problem is solved. The other extreme case is to assume that there is no ideal computer and the essential signals for control or compensating are unobtainable, the non-linearities, the noise problem and the limited equipments are such that the theory can not apply at all. In the former case there are many assumptions such as the system is linear or the non-linearily is confined to a certain type, but this is not the actual situation in real physical systems. The mathematical solution can only be used as a guide to design. In order to apply the theory to physical systems, some modifications usually have to be made. The general method is based on approximation. But how well the approximation can be designed is also a problem that needs to be considered. Since the theory of a discontinuous system had been generally discussed in this chapter, then some examples are needed to illustrate how well the theory can be applied. Also needed are some numerical examples to get a clear idea about the problems encountered in actually designing physical systems. The later chapters are written with this purpose. The approach used to solve problems later is closely related to the actual physical system with no attempt to use complex computers. This also can provide the idea of how to obtain (the best result from a poor situation). Of course, it is apparent that the better the theory which can be applied the better the results.

From the general block diagram in the first section, it is known that there are many possible ways to apply this discontinuous theory. Many authors have made various investigations along several branches of this discontinuous system theory. In the following chapters the investigations are made for several systems, they all have different characters. The general application and the modifications of the theory are given in each chapter whenever they are needed. There are many problems for which a complete solution cannot be given in these few chapters. In some cases it is possible only to point out the general ideas.

CHAPTER II - PHASE SPACE ANALYSIS AND DESIGN OF LINEAR DISCONTINUOUSLY DAMPED FEED BACK CONTROL SYSTEMS

Part I - General Description

SECTION I - SYNOPSIS

Linear systems can be designed to have very fast responses with essentially deadbeat performance by designing compensating loops which are switched in (and out) of the system as required. Fast response is obtained by using an unstable or nearly unstable uncompensated system to provide the desired rise-time. Deadbeat performance is obtained by designing the compensation loops to provide an overdamped system when the compensation is switched in. Selection of the real roots for the overdamped system is based on the desired location of the eigenvectors in the phase space. A switching computer is required which connects the compensation loops as the state point reaches a hyperplane which is related to the eigenvector in a special way. The computer is readily realized from derivative signals, and considerable engineering simplification is permissible in both the compensation design and the switching computer because of the topological nature of the phase portrait in the vicinity of properly chosen eigenvectors. Experimental results verify the theoretical conclusions.

SECTION II - THEORETICAL BACKGROUND.

It is clear from linear theory that a feedback control system with high gain has fast response and is accurate in both static and steady state. Unfortunately high gain is usually accompanied by light damping with consequent oscillatory transient response. Various investigators have proposed schemes which essentially vary the damping as some function of the error to obtain small ζ when the error is large and vice versa. Perhaps the simplest and most obvious approach is that of varying the gain of the main

amplifier as suggested by Blumenthal and Beck.[1] Another early and attractive proposal is that due to Lewis,[2] which uses a nonlinear signal to produce a smooth variation in γ throughout the range of operating inputs. Other proposals have suggested a discontinuous change in γ . Two such investigations have dealt with relay servos[3,4] where the damping loop was permanently connected but was operative only in the relay dead zone. Still others have been concerned with switching in a feedback loop. Flügge-Lotz[5] and Taylor propose to alter both position and velocity feedback in step fashion according to a predetermined schedule which is implemented by a fairly complex computer. Meiksin[6] proposes switching in the main feedback path from positive feedback for fast rise time to negative feedback for good damping. For second order systems switching along an eigenvector is easily instrumented, but a gain design suitable for overdamped dynamic response may not be compatible with static accuracy requirements. For third order systems eigenvector switching is recommended, the mathematical development of the switching criterion follow the matrix methods of Bogner and Kazda,[7] and lead to a rather complex switching computer, two switching operations being normal operation, though simplification of the computer and use of only one switching operation obtains under special conditions. Another discontinuous damping scheme is proposed by Ostrovskii,[8] who suggests switching in feedback paths to alter the coefficients of the closed loop differential equation. The results reported in this chapter with advantages as indicated, are in extension of this latter concept.

SECTION III - PHASE SPACE ANALYSIS

The differential equation of an Nth order linear feedback control system may be expressed as

$$A_0 E + A_1 \dot{E} + A_2 \ddot{E} + \cdots A_{n-1} \overset{n-1}{E} + A_n \overset{n}{E} = f(\dot{\theta}_R, \ddot{\theta}_R \ldots) \tag{2-1}$$

The characteristic equation has roots, which may be real, imaginary or complex depending on the values of the A's. Assume two sets of values for the coefficients, such that for one set there is a dominant pair of roots which are deliberately set near the imaginary axis to provide a suitable rise-time, and for the second set of coefficients all roots are real and negative. Since both equations are of the same order their phase portraits may be studied in the same phase space. For the underdamped case the singular point is a focus and the phase trajectories converge on this focus. When the system is stable all trajectories ultimately reach the origin of the error phase space providing:

1. The system is Type 1 and is excited by initial conditions and/or a step displacement.
2. The system is Type 2 and is excited by initial conditions and/or a step displacement and/or a ramp function, etc.

For the overdamped system (all real negative roots) the entire phase space is filled with trajectories which terminate at the origin (for the same conditions as above) and most of these trajectories exhibit a monotonic variation. It is well known that for each of the real roots there is an eigenvector, which corresponds to a phase trajectory that is a straight line. It is possible to construct hyperplanes such that each hyperplane corresponds to an eigenvector, and each hyperplane also contains a sub-set of complete phase trajectories. When the hyperplanes are chosen so that they contain a complete sub-set of phase trajectories, they subdivide the phase space into regions such that no phase trajectory can pass out

of one region and into another. Thus the hyperplanes act as boundaries which funnel the phase trajectories into the origin.

During operation of the system only one of the two equations can be effective at a given instant, but for purposes of analysis assume that both families of phase trajectories exist simultaneously. At every point in the space two trajectories (and only two) intersect, one curve being from each family. If a step displacement is applied to the underdamped system the state point follows a selected trajectory which intersects an infinite number of phase trajectories of the overdamped system. If the proper compensation circuits have been devised to change the coefficients of the characteristic equation from the selected underdamped case to the selected overdamped case, then the switch interserting these may be thrown at any time, and the state point transfers smoothly from the underdamped trajectory to the intersecting overdamped trajectory. To prove this it is only necessary to note that the coordinates of the state point at the switching instant are $P_S = (E_S, \dot{E}_S, \ddot{E}_S, \text{--------} \overset{n-1}{E}_S)$. Now this point lies on both phase trajectories and thus satisfies both characteristic equations, all derivatives being continuous except the nth, in which discontinuity is permissible. Note that this condition obtains for any arbitrarily selected switching point, and as a result a transientless switching operation is always obtained.

To provide automatic switching at the preselected point, note that a straight line through the origin of the phase space and through the selected switching point has direction numbers $E_S, \dot{E}_S, \ddot{E}_S, \text{----} \overset{n-1}{E}_S$ and the symmetric equations of this line are

$$\frac{E}{E_S} = \frac{\dot{E}}{\dot{E}_S} = \frac{\ddot{E}}{\ddot{E}_S} \text{---------} = \frac{\overset{n-1}{E}}{\overset{n-1}{E}_S} \tag{2-2}$$

This is not a convenient form for implementation of a switching computer, so it is noted that a hyperplane containing this line is given by

$$a_0 E + a_1 \dot{E} + a_2 \ddot{E} + \dots a_{n-1} \overset{n-1}{E} = 0 \tag{2-3}$$

where the coefficients a_0, a_1, etc. are chosen so that

$$a_0 E_s + a_1 \dot{E}_s + a_2 \ddot{E}_s + \dots a_{n-1} \overset{n-1}{E_s} = 0 \tag{2-4}$$

Equation(2-3)indicates that the only switching computer needed is an adding amplifier, providing all of the derivative signals are available or can be generated. Such a switching computer is sensitive to all of the points on the hyperplane defined by equations(2-3) and it is necessary that this hyperplane be carefully chosen if specified performance is to be obtained. When a step displacement input is the only input to be considered, only those phase trajectories starting on the error axis are important. Assume that a switching point on one of these step response trajectories is chosen to define equations (2-2), (2-3), (2-4). All other step trajecteries pierce the hyperplane of equation (2-3) along a straight line through the origin which may be considered a mapping of the E-axis in the hyperplane. Thus for step displacement inputs only a line in the hyperplane is actually used for switching, and any other hyperplane which contains this line may be instrumented and used if more convenient. Note that while the step response is insensitive to the hyperplane chosen, the response to other inputs is affected. It should also be noted that a computer designed to implement equation (2-3) produces an output the sign (or polarity) of which depends on the sign or direction of the step input. Then for use with a polarity sensitive relay a satisfactory switching equation in

$$\dot{E}(a_0 E + a_1 \dot{E} + a_2 \ddot{E} + \dots a_{n-1} \overset{n-1}{E}) = 0 \tag{2-5}$$

Several practical modifications of this are discussed in a later section.

The system is discontinuously damped because the operation of the switch circuits which change the roots from complex values to real, negative values. Nevertheless the system may be considered linear for step displacement inputs. Note that the switching hyperplane subdivides the phase space into two regions, and in each region the system is completely linear but in each region a different linear differential equation applies. However, the switching is a straight line through the origin of the phase space, and this insures that the initial conditions after switching are always directly proportional to the magnitude of the step. This leads to the following characteristics:

a) The rise time to the switching point is the same for all magnitudes of step.

b) The settling time of the composite system is constant.

c) The percent overshoot (if any) is constant.

These features are independent of the choice of a switching line, but the numerical values associated with each depend very much on this choice. Since the purpose of discontinuous damping is to permit a very fast rise time with deadbeat (or nearly deadbeat) response and a minimum settling time, the following considerations are important: for fast rise time switching should be delayed until E is as near to zero as is permissible; for deadbeat response due to the heavily damped system, a trajectory must be selected which is nearly a straight line to the origin (i.e., a fast eigenvector if this is possible) since departure from this choice leads to the long-tailed response (long settling time) characteristics of overdamped systems.

For second order systems switching on the eigenvector is possible and practical. For higher order systems the trajectory of the underdamped system selected by the step displacement input does not pass through that

line in the phase space which is the desired eigenvector and therefore the optimum trajectory is not available. However, for each eigenvector there exists a hyperplane such that a sub-set of trajectories lies entirely in this hyperplane and these trajectories become tangent to the eigenvector at the origin. In general the equation of this hyperplane is known to be one order lower than the characteristic equation, and is formulated by simply removing the root corresponding to the eigenvector. Thus

$$\overset{n-1}{E} + \sum_{i=2}^{i=n} r_i \overset{n-2}{E} + \ldots + \prod_{i=2}^{i=n} r_i E = \tag{2-6}$$

is the equation for the hyperplane for removed r_1. It is also the defining equation for the switching computer needed to introduce the discontinuous damping.

Fig. 2-1 shows two phase portraits on the E vs. $\dot{E}$ plane for an overdamped third order system. Note that in both cases the three eigenvectors can be located, and are straight lines as expected. Fig. 2-1a was obtained such that all of the trajectories around the eigenvectors lie only in the hyperplanes associated with that eigenvector. For Fig. 2-1b the initial conditions were chosen such that the trajectories are in the vicinity of the hyperplane but not actually in it.

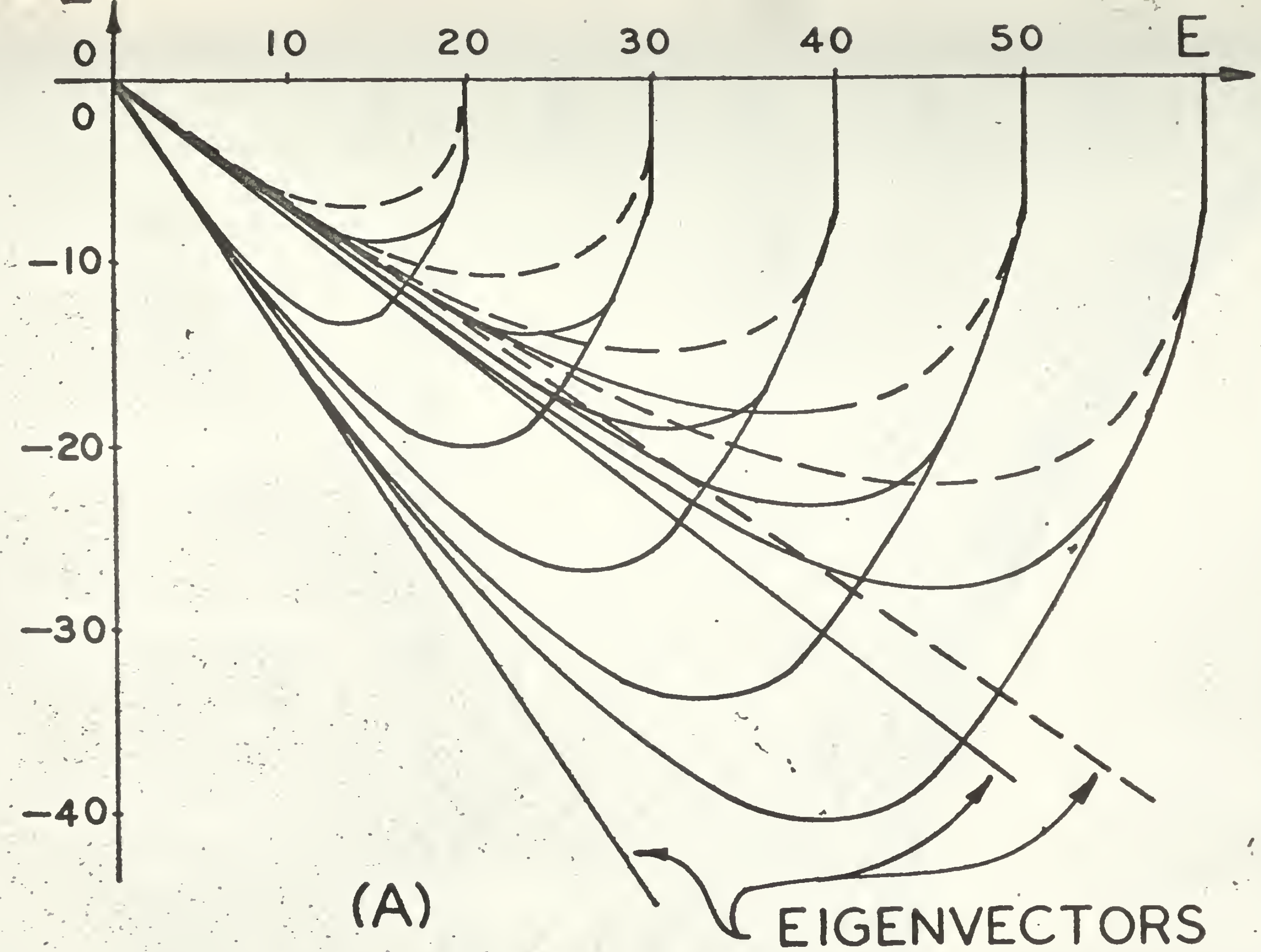


Fig. 2-1 Phase Plane Plots of a Third Order System with Discontinuously Damped Feedback.
(A) Switching in the Hyperplane.

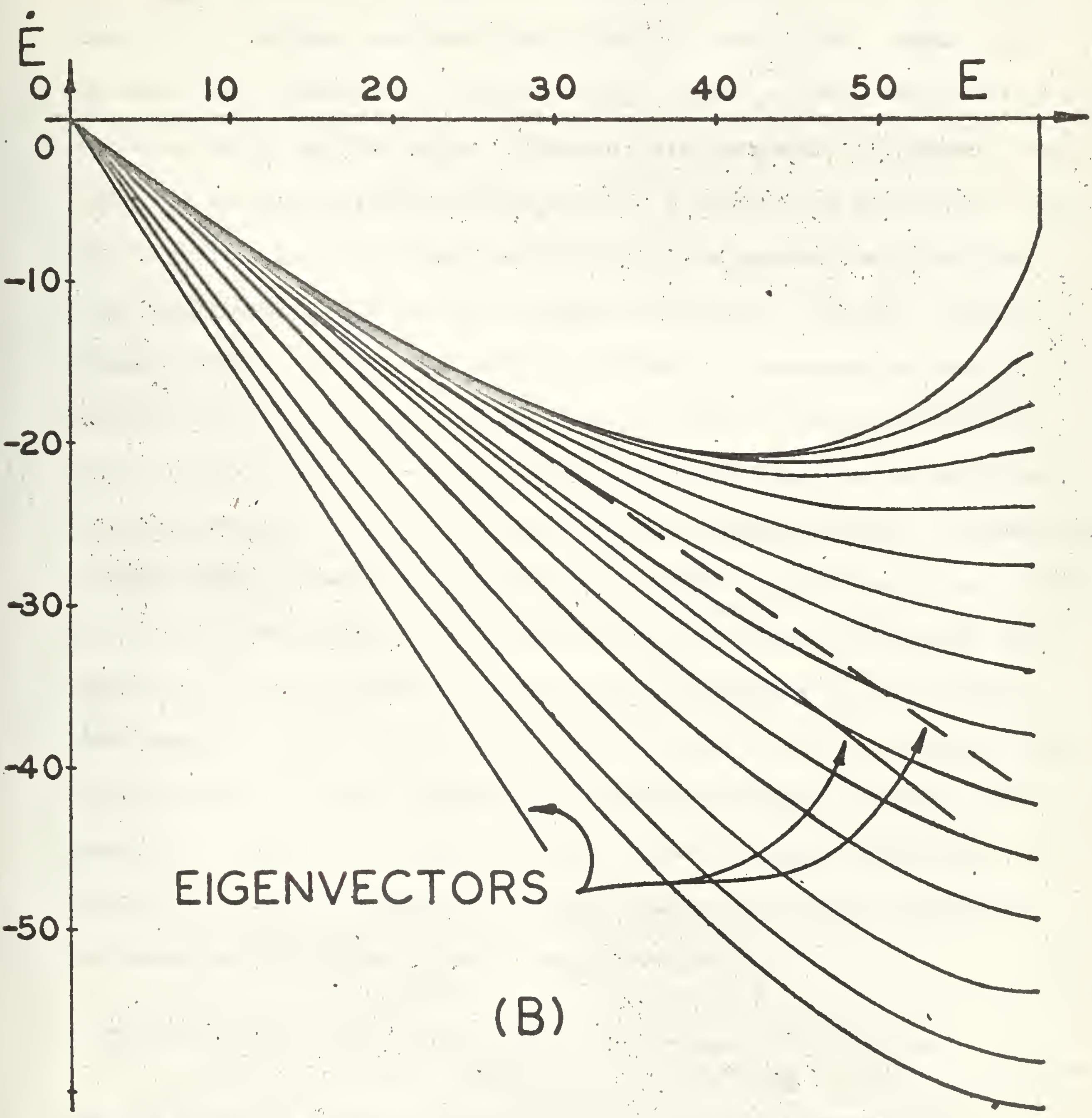


Fig. 2-1 (B) Switching not in the Hyperplane.

SECTION IV - THEORETICAL CONSIDERATIONS

In the design of a discontinuously damped system of this type, three theoretical problems (and some other practical ones) arise. These are the design of the uncompensated and compensated linear systems, the selection of the hyperplane, and the design of the switching computer. In general the design of the uncompensated system would be a single loop design with gain set to satisfy both the steady state accuracy requirements and the rise time requirements. The resulting system is expected to be very lightly damped, perhaps unstable. As far as the theory is concerned an unstable system is permissible, in practice a stable system is more desirable because of the possibility of a failure in the switching loop, so some fixed compensation may be used. The design of the overdamped system is accomplished very simply by arbitrarily selecting a suitable set of real roots, evaluating the coefficients of the corresponding characteristic equation, then computing the gain constants for the various derivative signal channels which must be used as compensaters. For the case in which the input signal is restricted to a step displacement input the derivatives of error are numerically equal to the derivatives of the output signal except for a factor of -1.0, so the generalized block diagram is as shown in Fig. 2-2. The equations are (assuming that G(s) has no zeros).

$$G(s) = \frac{E}{s^n + (\gamma_1 + \gamma_2 + \cdots)s^{n-1} + \cdots + (\gamma_1\gamma_2\gamma_3 \cdots)s^x} \tag{2-7}$$

and the composite feedback function when the switch is closed is

$$E(s) = 1 + As + Bs^2 + \cdots + Ns^n \tag{2-8}$$

where $n > N$, and normally $n \doteq N+1$.

Then

$$G(s)H(s) = \frac{K(1 + As + Bs^2 + \cdots + Ns^n}{s^n + (\gamma_1 + \gamma_2 + \cdots)s^{n-1} + \cdots + (\gamma_1\gamma_2\gamma_3 \cdots)s^x} \tag{2-9}$$

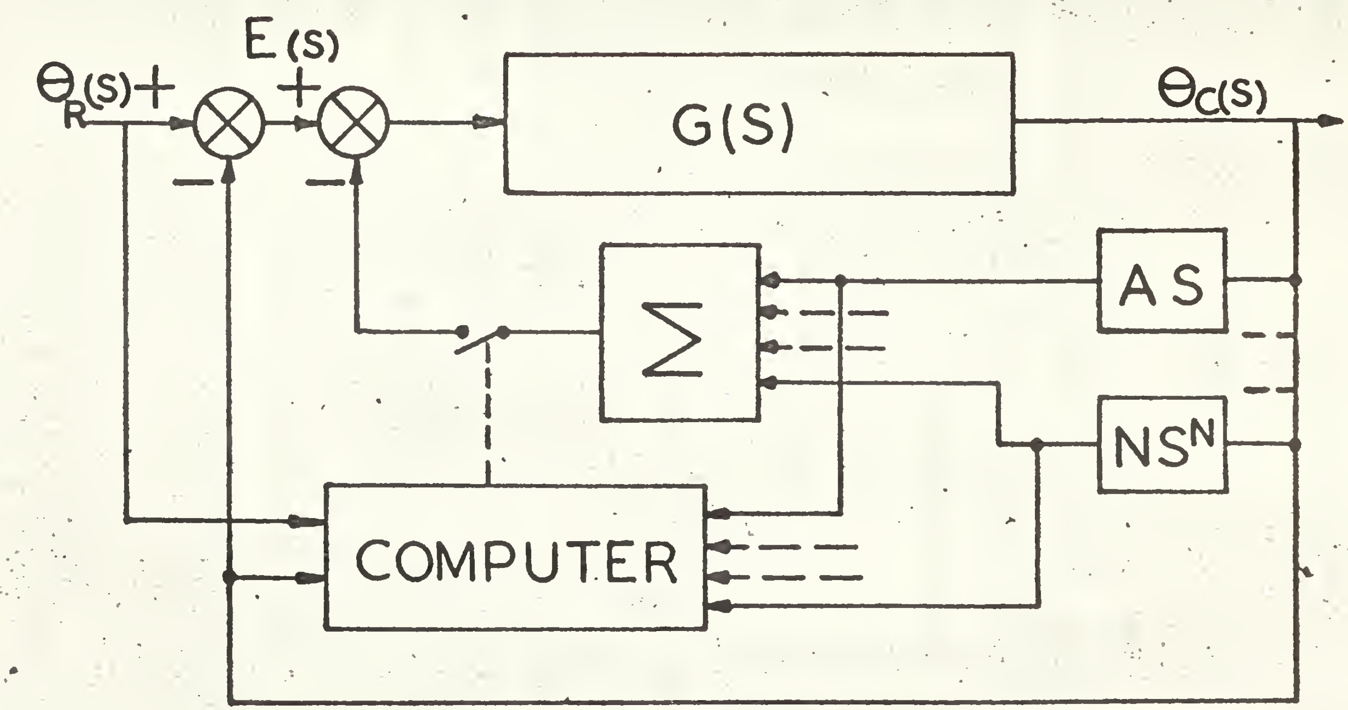


Fig. 2-2 Generalized Block Diagram for nth Order System with Discontinuous Feedback Compensation.

and if $n = N+1$ the characteristic equation becomes

$$0 = s^n + (\Upsilon_1 + \Upsilon_2 + \cdots + KM)s^{n-1} + \cdots (\Upsilon_1\Upsilon_2\Upsilon_3 + KX)s^x + \cdots KBs^2 + KAs + K \tag{2-10}$$

For a Type 1 system $X = 1$ and the characteristic equation becomes

$$0 = s^n + (\Upsilon_1 + \Upsilon_2 + \cdots + KM)s^{n-1} + \cdots + (\Upsilon_1\Upsilon_2\Upsilon_3 \cdots + KA)s + K \tag{2-11}$$

All Υ 's are known from the uncompensated system, K is known, and each desired coefficient is known from the arbitrary selection of the real roots, thus each coefficient in equation(2-11)may be equated to the corresponding coefficient for the desired overdamped system and A, B, C, ...N are evaluated. The choice of real roots is based on a desire to obtain rapid damping of the response during the overdamped mode of operation. To obtain this, one and only one of the real roots is chosen quite small, all others are chosen quite large. This combination gives rise to an eigenvector which projects on to the E vs. $\dot{E}$ plane at a relatively small angle to the $\dot{E}$-axis, and the hyperplane associated with this eigenvector is used as the switching surface. When this is done the initial conditions at the switching instant cause the residue at the smallest root to go to zero, thus the settling time is controlled solely by the large roots and as a consequence is quite small. To show that the smallest real root does not appear in the transient response after switching, consider a third order system for which the differential equation is

$$\dddot{E} + (r_1 + r_2 + r_3)\ddot{E} + (r_1 r_2 + r_1 r_3 + r_2 r_3)\dot{E} + r_1 r_2 r_3 E = 0 \tag{2-12}$$

and the real roots are $r_1 < r_2 < r_3$.

Equation(2-12)has a solution

$$E(t) = \beta_1 e^{-r_1 t} + \beta_2 e^{-r_2 t} + \beta_3 e^{-r_3 t} \tag{2-13}$$

where

$$\beta_1 = \frac{r_2 r_3 E_S + (r_2 + r_3)\dot{E}_S + \ddot{E}_S}{(r_2 - r_1)(r_3 - r_1)} \tag{2-14}$$

E_s, $\dot{E}_s$ and $\ddot{E}_s$ are the initial conditions and β_2 and β_3 are not relevant to this discussion. Now the hyperplane at which switching occurs is chosen to be:

$$r_2 r_3 E + (r_2 + r_3)\dot{E} + \ddot{E} = 0 \tag{2-15}$$

therefore, when switching occurs at any point in the hyperplane the initial conditions at that point force the numerator of equation (2-14) to zero, and the $\beta_1 e^{-r_1 t}$ term disappears from equation (2-13). This is true for any order equation. For a third order system such switching reduces the response to a second order response, thus motion is in the hyperplane. For higher order systems motion is confined to a subspace of order N-1. Since every point on the resulting phase trajectory satisfied equation (2-6), the switching computer output remains at zero and the switch has no tendency to reopen.

The function of the computer is to operate the switch when the state point reaches the selected hyperplane. Thus equation (2-6) must be mechanized, which is relatively simple since it may be assumed that all derivative signals have been made available in providing the compensation loop. It only remains to scale the magnitudes of these signals to provide the coefficients defined by the chosen roots of the overdamped system. This can usually be done with potential dividers, then the signals are summed to formulate equation (2-6), and the summation signal feeds into a circuit which operates a relay when the signal goes to zero. For a polarity sensitive relay circuit to recognize the difference between positive and negative steps, one solution is to multiply equation (2-6), by $\dot{E}$. This complicates the computer machanization, however, and for many applications other methods

may be substituted. For an nth order system, equation (2-6) is the equation of the hyperplane which corresponds to the eigenvector and thus is the defining equation for the switching computer. If this hyperplane is used as a switching surface for step displacement inputs, each phase trajectory starts on the E-axis and pierces the hyperplane at some point which may be called a switching point. The loci of these switching points is a line on the surface of the hyperplane. Furthermore it is a straight line which passes through the origin. Note that only the points on this line are used in switching. Therefore, if any other hyperplane can be instrumented in the switching computer such that the new hyperplane intersects the switching hyperplane along a trace which is exactly the switching line, then for step displacement inputs the operation of the new switching hyperplane is exactly correct. Since the switching line is a straight line through the origin, its projection onto the E vs. $\dot{E}$ plane is also a straight line through the origin. The equation of this line is readily computed and is of the form

$$E + A\dot{E} = 0 \qquad (2\text{-}16)$$

A switching computer to instrument this is a very simple adder, and because of the simplicity the coefficient A may be set by adjustment rather than calculation. The surface thus defined is a hyperplane which (in the three dimensional case) is perpendicular to the E vs. $\dot{E}$ plane. When switching occurs in response to a step input, the state point does not remain in the surface defined by equation (2-16), but in the surface defined by equation (2-6). Thus the switching computer develops an output and the relay will reoperate unless proper precautions are taken in the design of the switching computer.

SECTION V - ANALYTIC AND COMPUTER VERIFICATION

Second Order System.

Consider the block diagram of Fig. 2-3a with transfer function as given in the blocks.

The differential equation of the system without compensation is

$$\ddot{E} + 2.6\,\dot{E} + 64\,E = 0 \tag{2-17}$$

for which $\zeta = 0.162$, $\omega_n = 8$. For the compensated system with $H = 0.3$ the equation is

$$\ddot{E} + 21.8\,\dot{E} + 64\,E = 0 \tag{2-18}$$

for which the real roots are at -18.3 and -3.5. For $H = 0.96$

$$\ddot{E} + 64\,\dot{E} + 64\,E = 0 \tag{2-19}$$

and the real roots are at -0.995 and -63.0. For the second order system the eigenvectors of the overdamped equations are in the E vs. $\dot{E}$ plane and thus can be used as switching lines. The two eigenvectors can be located easily since they are precisely those isoclines for which the slope of the isocline is identical with the slope of the trajectory and also is the value of the real root. To illustrate this Fig. 2-3b shows an isocline plot for equation (2-18) and Fig. 2-3c shows an isocline plot for equation (2-19). Typical trajectories for operation as a discontinuous system have been constructed and it is apparent that slight inaccuracies in switching do not cause a significant difference.

This system was simulated in the analog computer as shown on Fig. 2-4. The switching line is instrumented very simply as a summer amplifier operating a normally closed two position relay. When the step is applied the relay automatically opens and remains open until the eigenvector is reached, at which point the relay voltage reduces to zero and the relay drops out, closing the damping circuit. For all points on the eigenvector this voltage sum remains zero and the relay is not actuated. The relay is sensitive

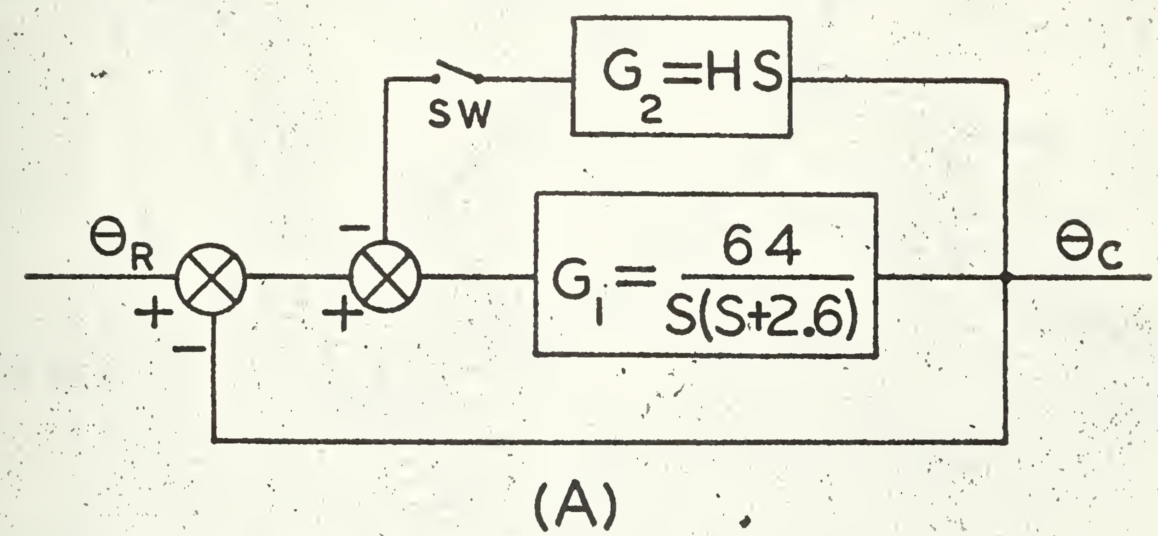


Fig. 2-3 Block Diagram and Phase Plane Plots of a Second Order System with Discontinuous Tachometer Feedback (For detail see Part II)

(A) Block Diagram.

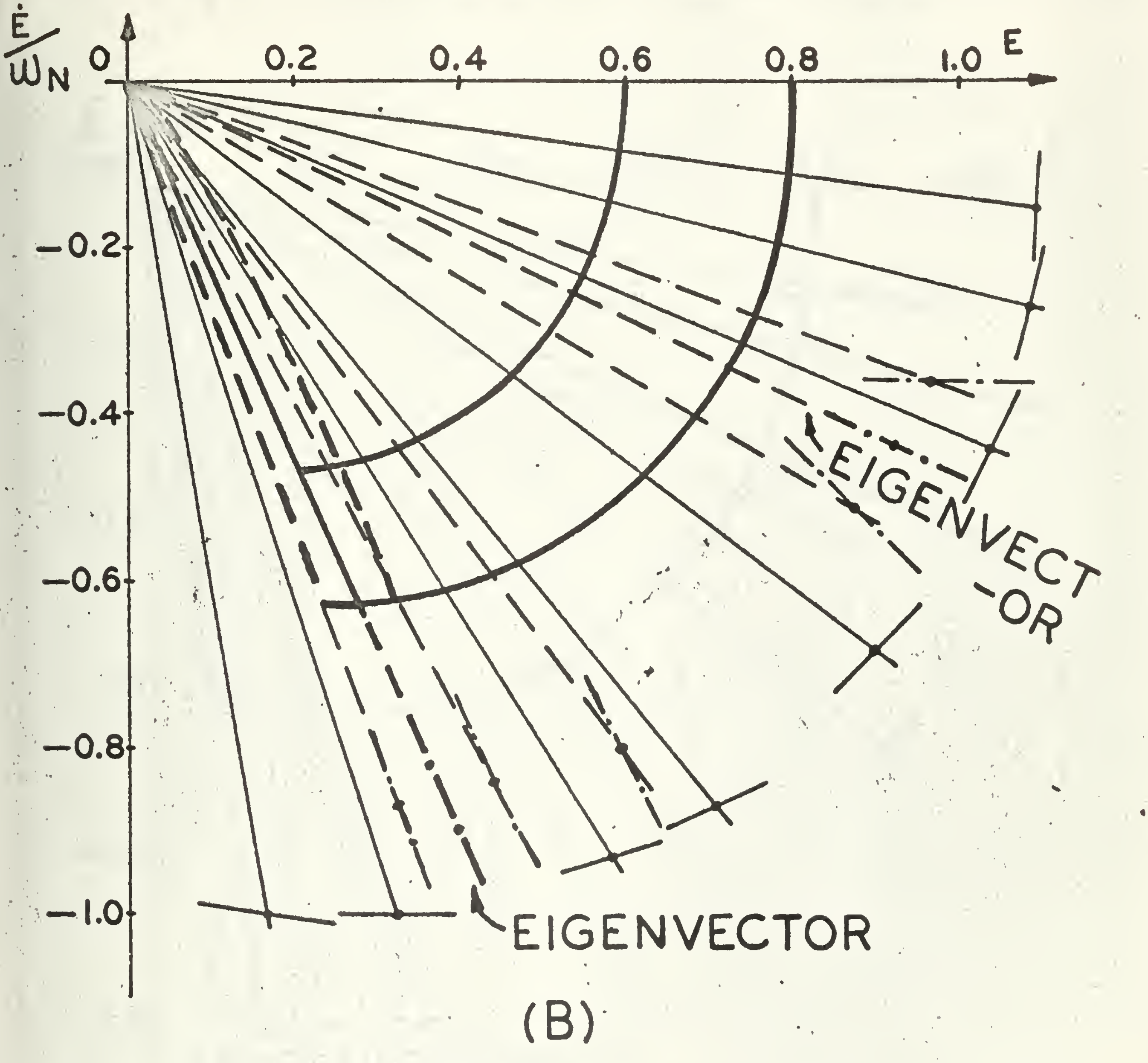


Fig. 2-3 (B) With Moderate Tachometer Feedback.

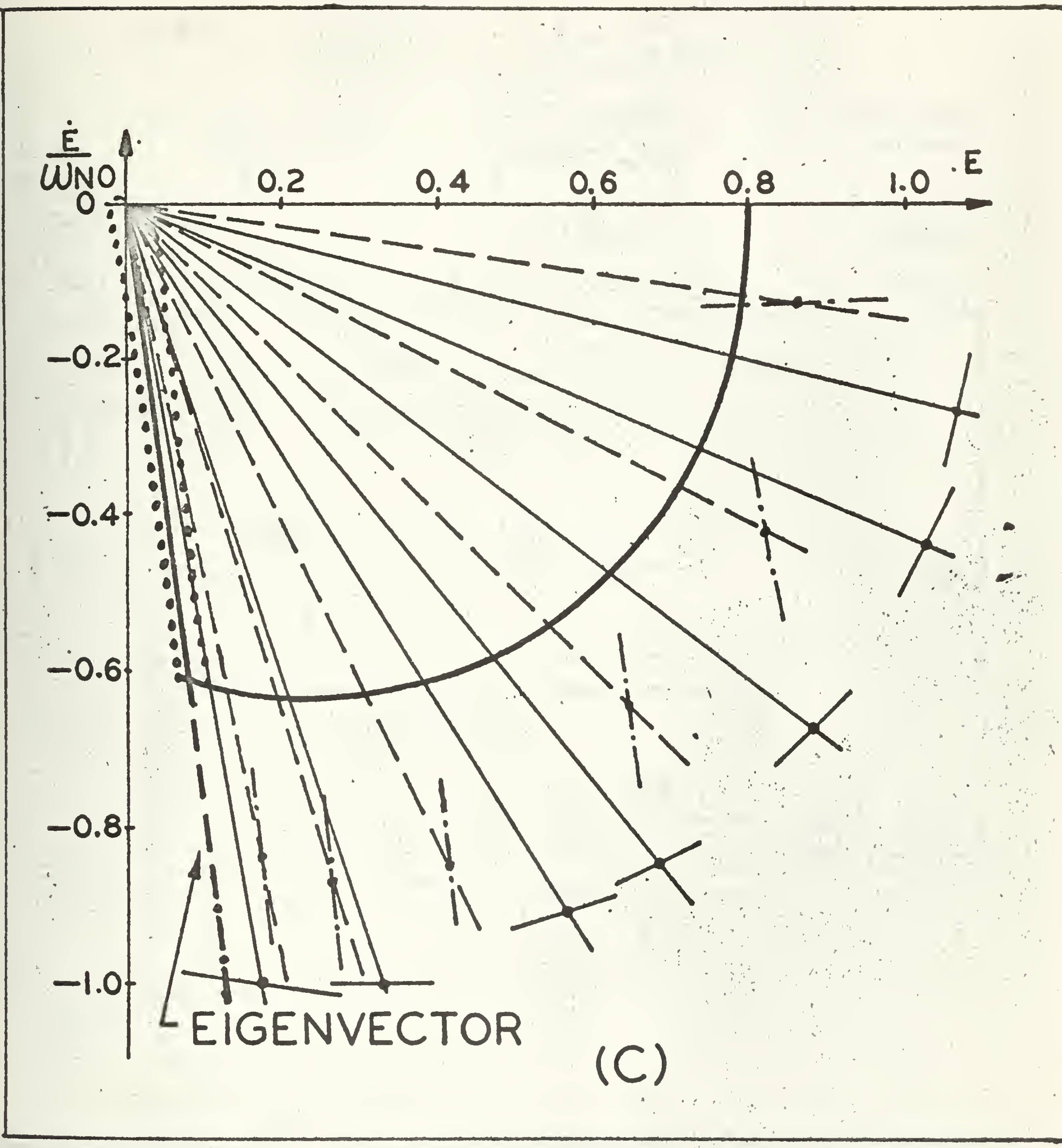


Fig. 2-3 (C) With large Tachometer Feedback.

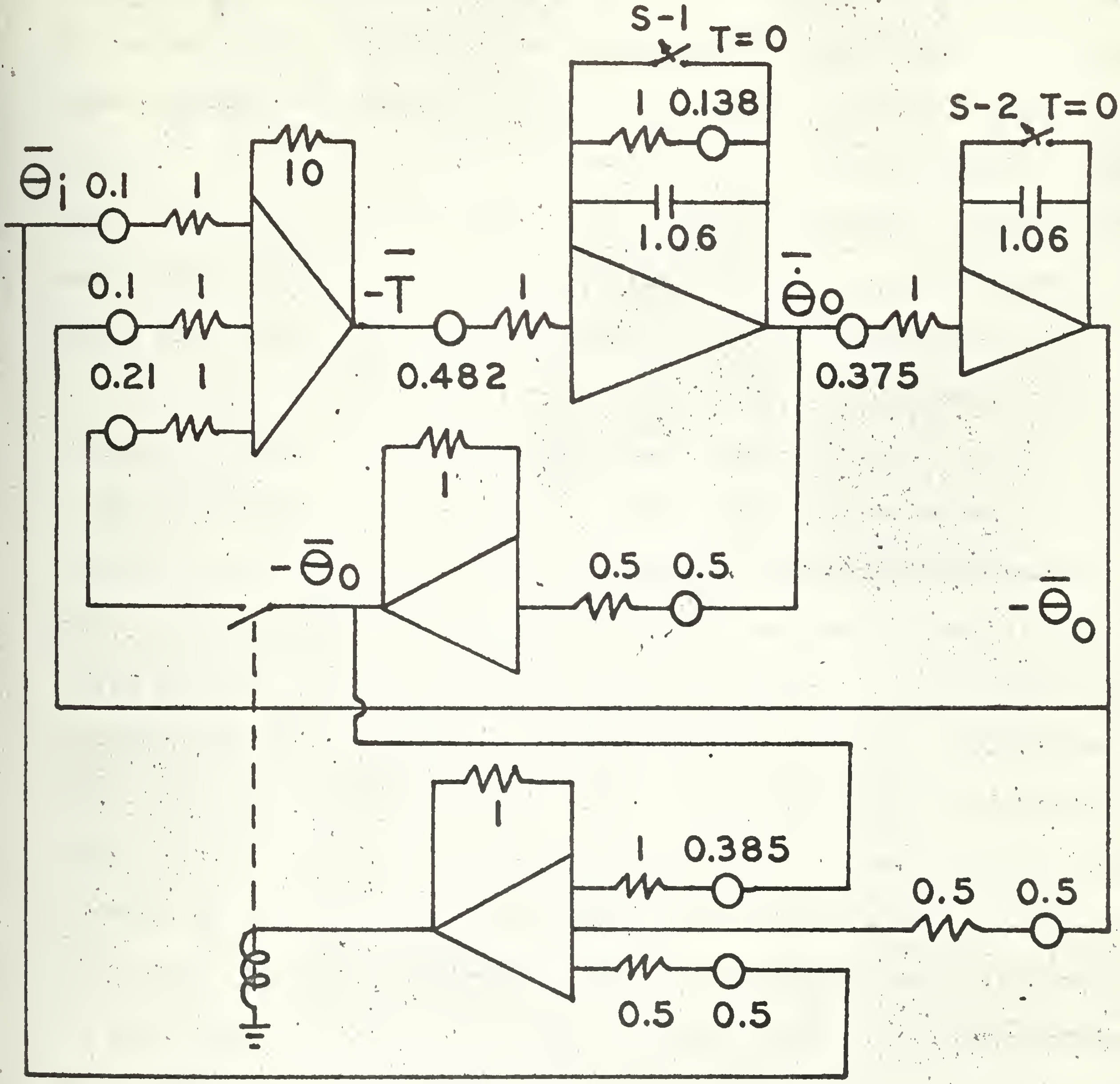


Fig. 2-4 Analog Computer Set up for a Second Order System with Discontinuous Tachometer Feedback. (For detail see Part II).

only to magnitudes, so no additional devices are required to distinguish between positive and negative steps. Fig. 2-5a compares the step responses of the underdamped, overdamped and discontinuously damped systems. Fig. 2-5b shows the transient responses obtained with various magnitudes of step input, and Fig. 2-5c shows the effect of variations in the switching computer adjustment. The slope of the switching line is readily adjusted by changing the magnitude of the velocity signal fed to the computer. Fig. 2-5c indicates that a wide range of adjustment is permissible without significant changes in the response. This, however, is due more to the characteristics of the switch and computer than to the phase plane topology. Curve 1 in Fig. 2-5c is the eigenvector trajectory and the system operates as predicted, with no additional operations of the relay. For earlier switching the trajectory should seek the slow eigenvector and thus should be concave upward. Actually the deviation of the trajectory from the switching line caused the relay to reopen repeatedly so that the trajectories shown are due to the system chattering down the switching line. For late switching the trajectories in this case are so nearly straight that the relay did not reoperate until the switching delay produced curve 4 and reversion to the underdamped condition then caused a tremendous overshoot. Thus this particular choice of a switching scheme permits early switching over as wide a range as desired providing relay chatter is acceptable. The actual trajectory obtained with relay chatter provides a faster response than would be obtained if it were possible to follow the overdamped trajectory from the same initial switching point. Late switching is obviously dangerous in this case. The width of the zone permissive for late switching depends on the sensitivity of the relay-switching computer unit, and if the gain of this circuit is high late switching may not be acceptable at all because of the possibility of reoperation of the relay causing a

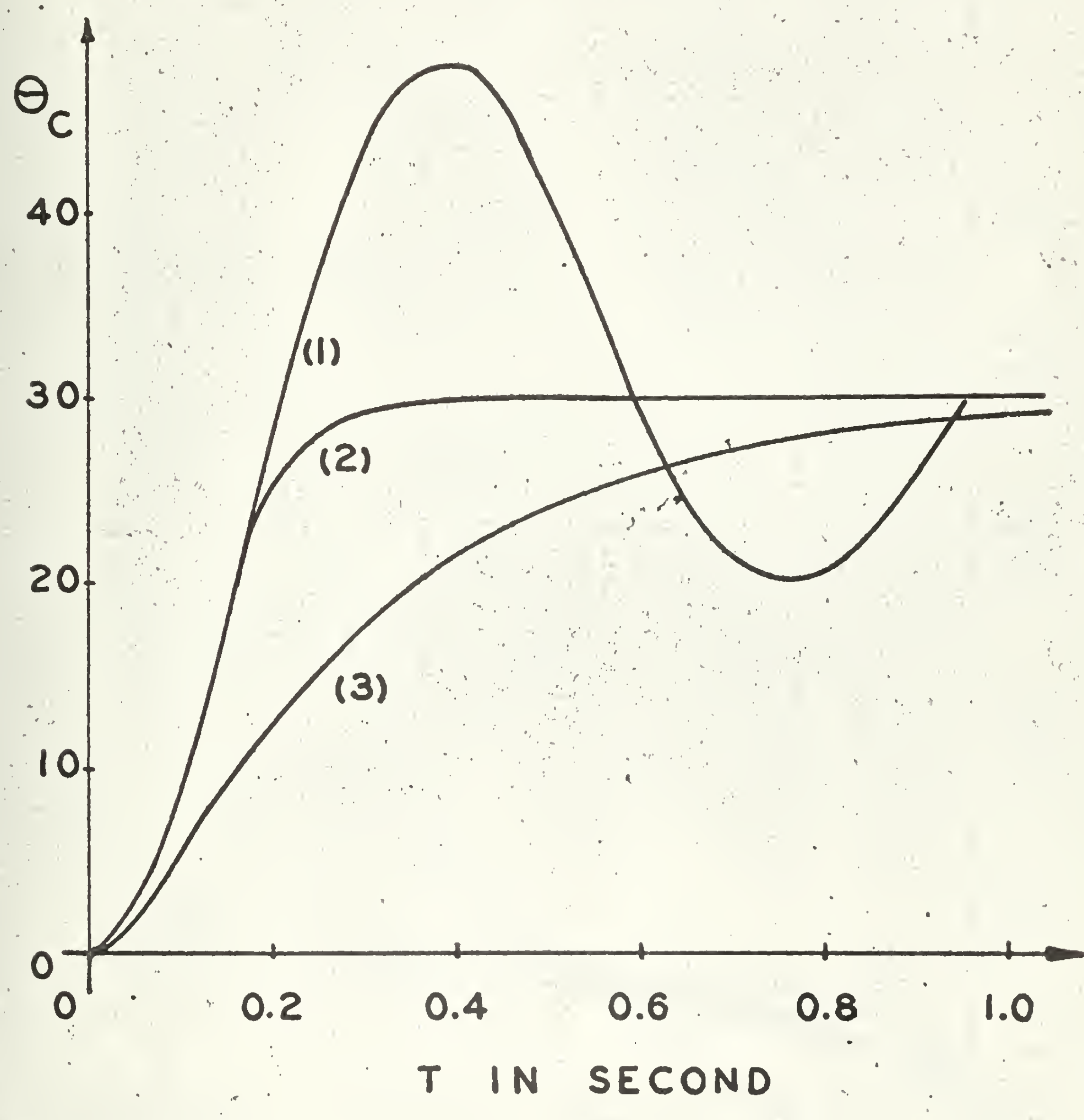


Fig. 2-5a Analog Computer Plots for a Second Order System with Discontinuous Tachometer Feedback. (1) Underdamped, (2) Discontinuously Damped, (3) Overdamped.

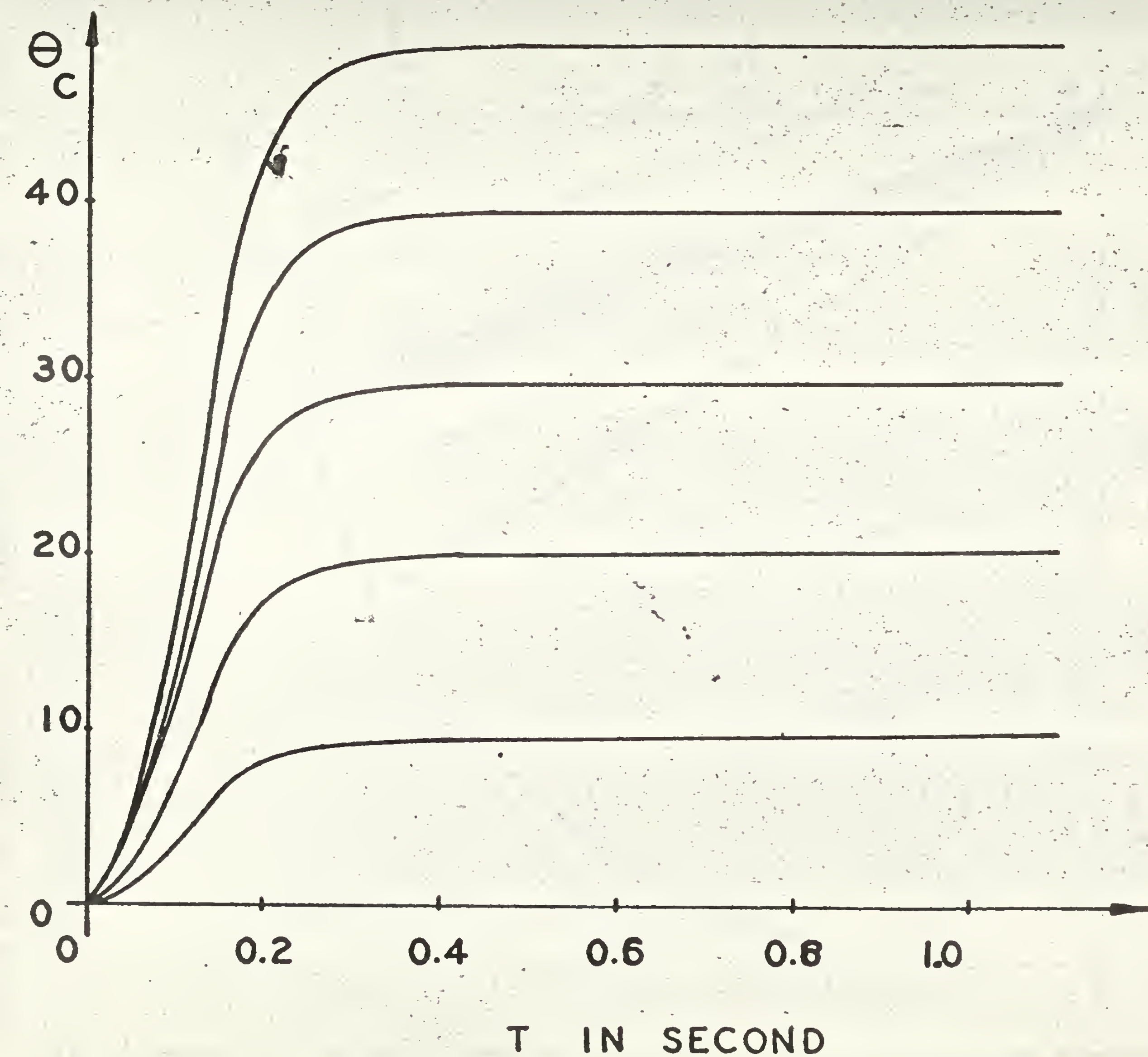


Fig. 2-5b Transient Response for Various Magnitudes of Step Input.

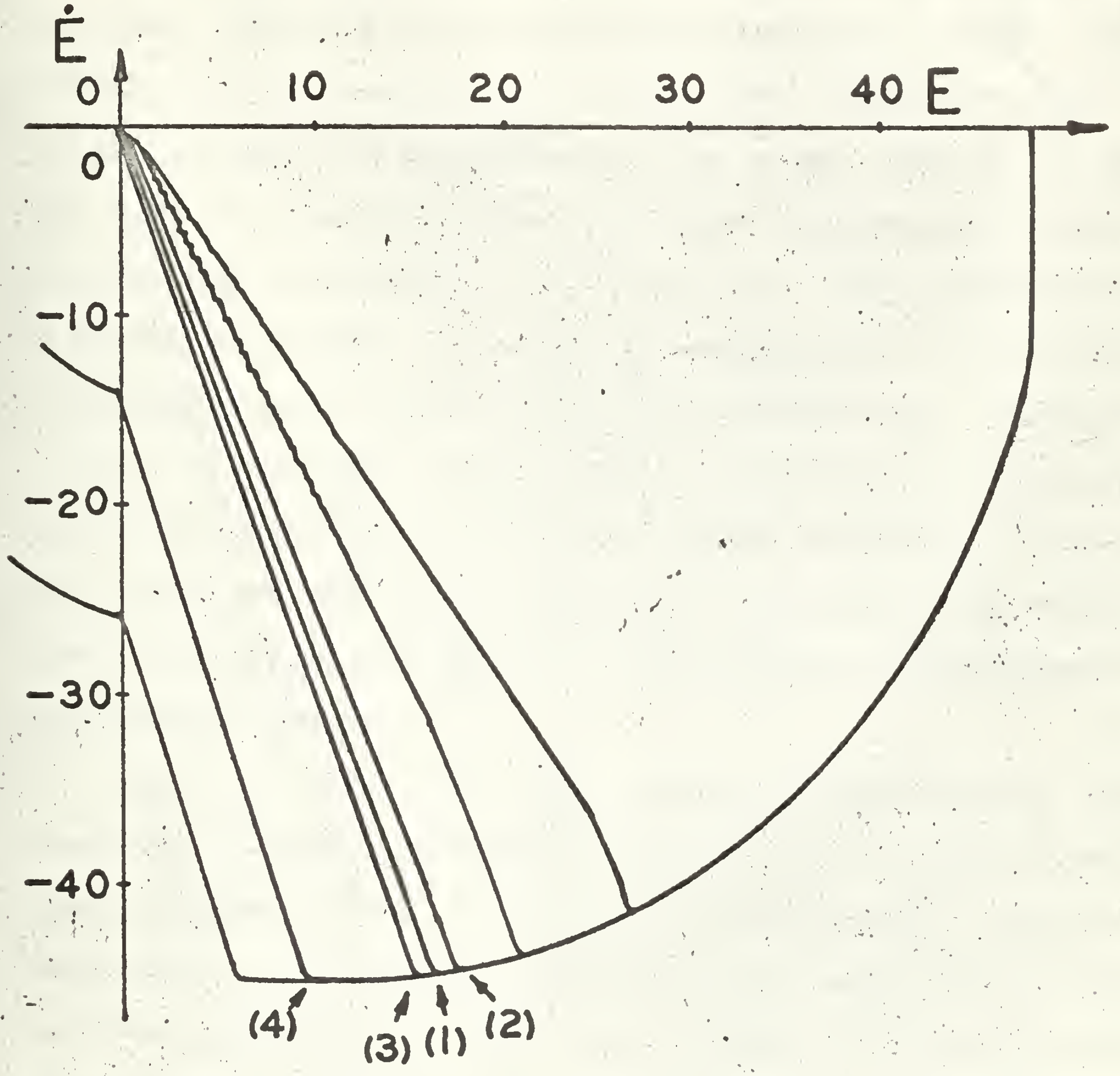


Fig. 2-5c Illustration of the effects of Variations in the Switching Computer Adjustment.
(1) Switching on the eigenvector, (2) Switching early, (3) Switching later, (4) Switching too late.

Fig. 5c

large overshoot.

Third Order System.

The block diagram of Fig. 2-6a shows the system, including transfer functions. The roots of the uncompensated system are at -0.180; -0.0006 $\mp$ j0.0625. For the compensated case the roots were chosen at -0.0115; -0.178; -0.354; from which the gains required in the feedback path are H = 44.5; M = 15.8. Fig. 2-6b shows the analog computer implementation including the switching circuit which is, once more, just a summer amplifier driving a normally closed relay. Note that two techniques were used to obtain the required $\ddot{\theta}_c$ signal, a normal differentiator-amplifier, and a simple R-C circuit. This was done for two reasons: practical applications normally require the simplicity of the R-C approximation, and thus it is important to know whether the added pole disturbs either the damping or the switching. Then, too, a performance comparison between the exact and approximate systems seemed necessary.

Fig. 2-7a compares the transient responses of the overdamped, underdamped and discontinuously damped systems, and Fig. 2-7b compares the frequency responses. On Fig. 2-7b curve (1) is the frequency response of the underdamped system, and also of the discontinuous system when the relay or switch has negligible dead zone. (With sinusoidal input all steady state signals vary periodically and equation (2-6) is satisfied for infinitesimal time on each cycle, so the damping is inserted for essentially zero time.) When the relay has small but appreciable dead zone the discontinuous damping is quite effective and the frequency response is curve (2) for a wide range of signal amplitudes. In the specific case tested for a range of test signal amplitudes of 0-50 volts (above 50 volts amplifiers saturated) curve (2) was obtained for all amplitudes from 7 to 35 volts. For amplitudes

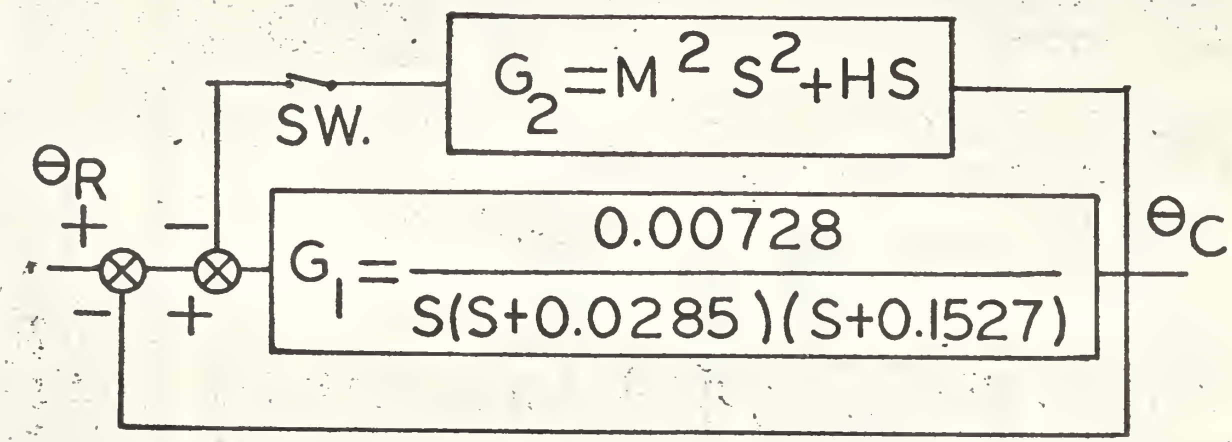


Fig. 2-6a Block Diagram of Third Order System

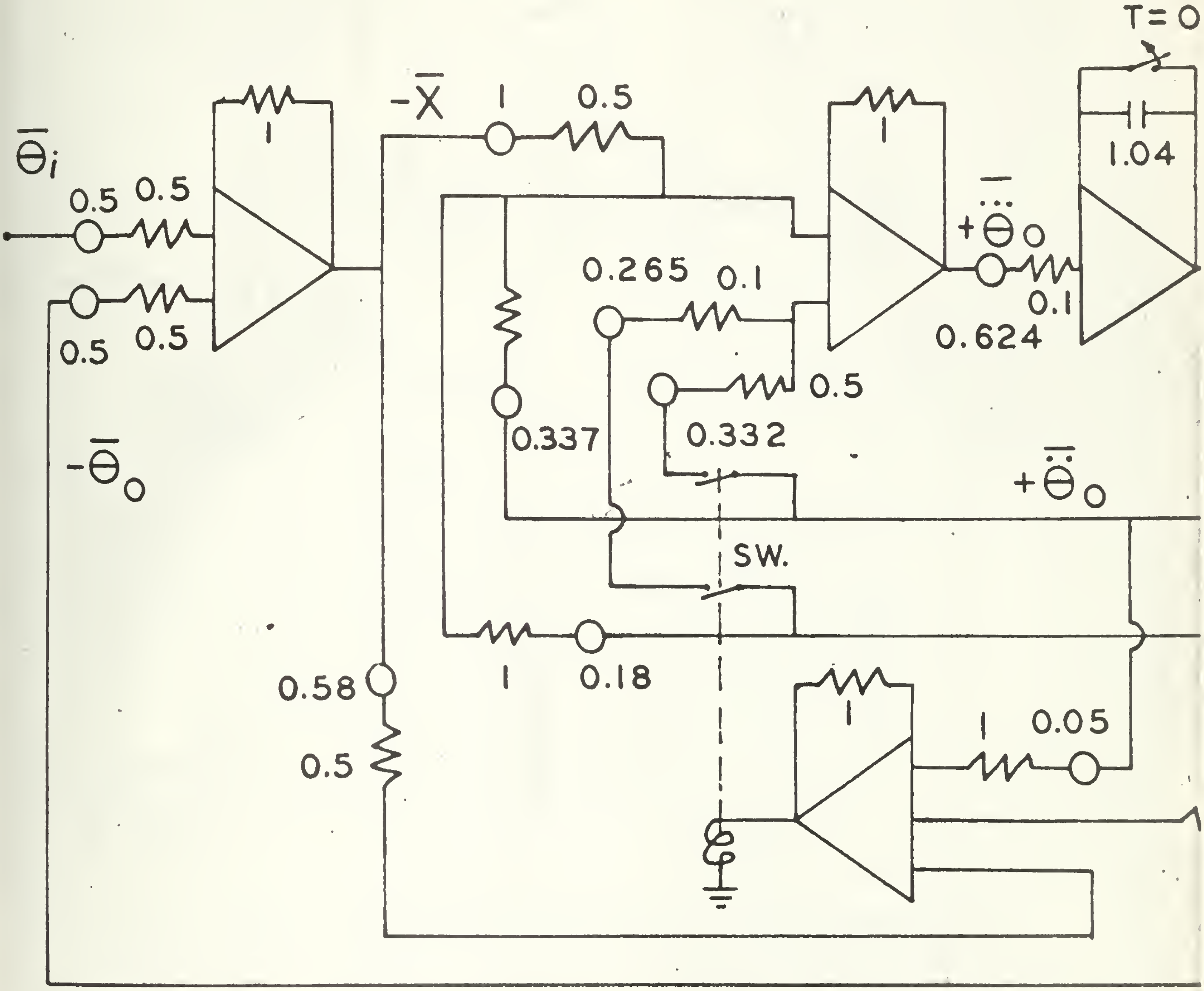


Fig. 2-6b Analog Computer set up for a third order system. (For detail see Section III (B) in Part II)

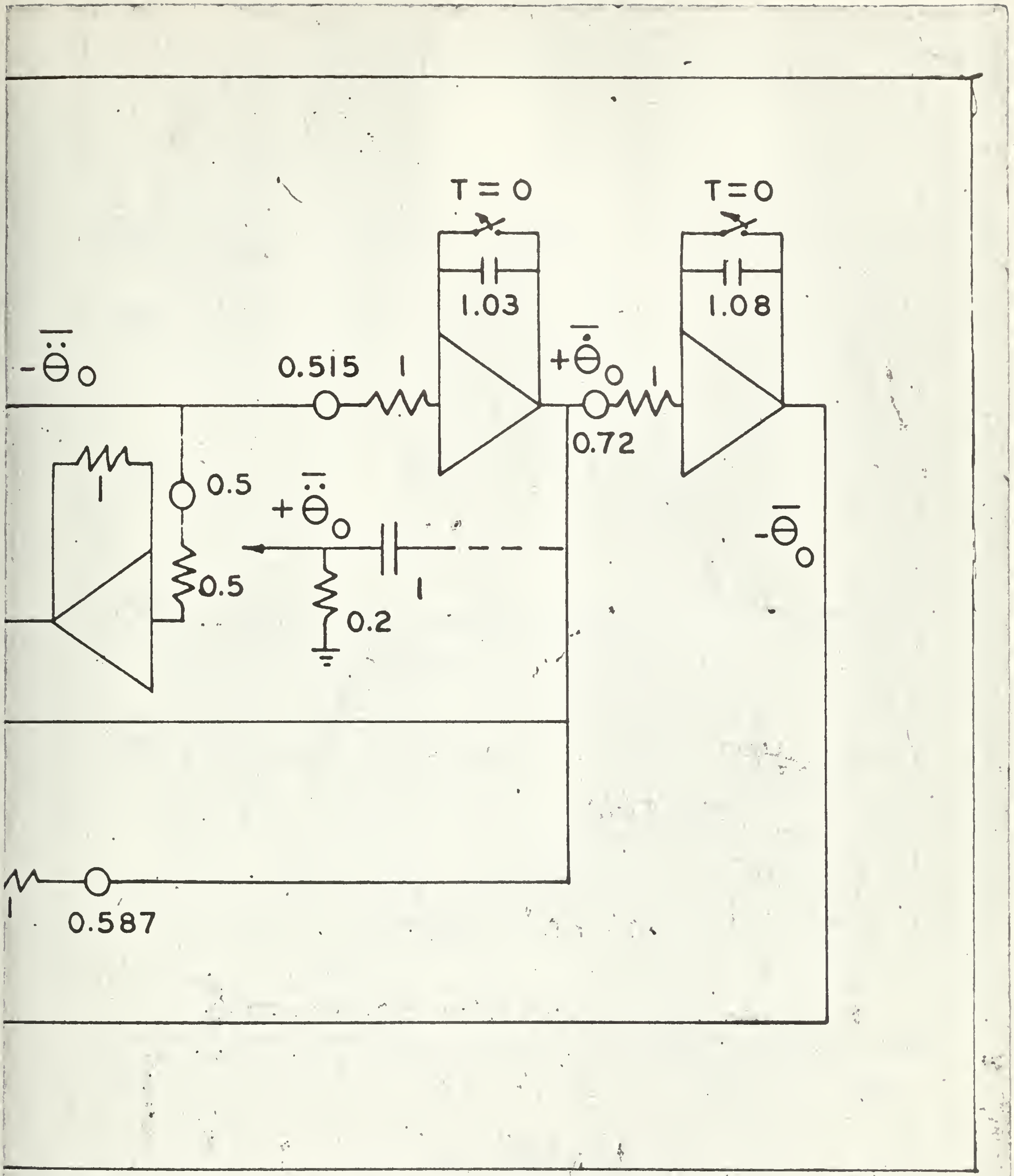

T = 0
T = 0
1.03
1.08
$-\bar{\ddot{\Theta}}_o$
0.515
1
$+\bar{\dot{\Theta}}_o$
1
0.72
1
0.5
$+\bar{\ddot{\Theta}}_o$
0.5
1
0.2
$-\bar{\Theta}_o$
1
0.587

Fig. 2-6b

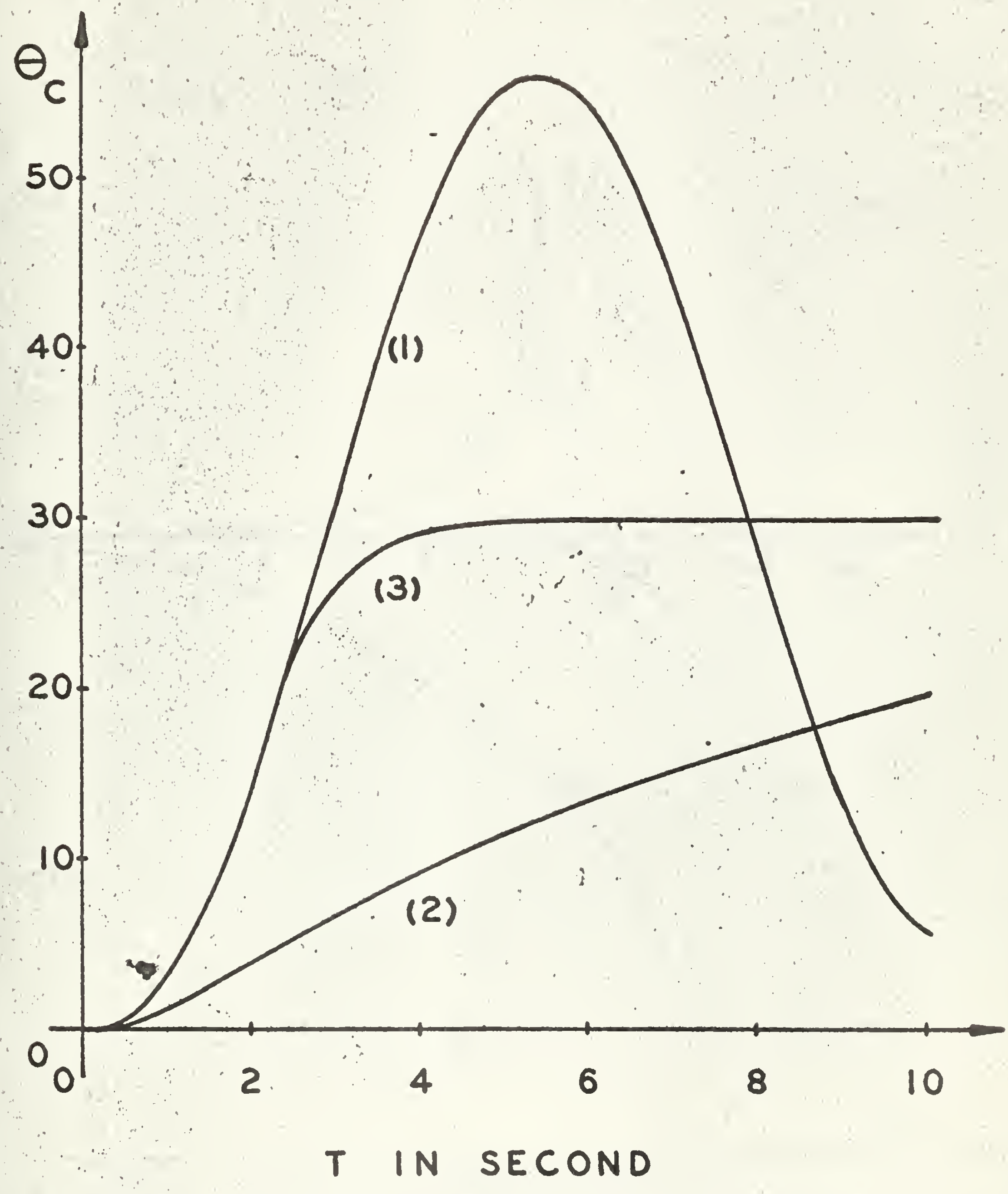


Fig. 2-7a Third order system transient response for a step input.
(1) Underdamped, (2) Overdamped, (3) Discontinuously damped.

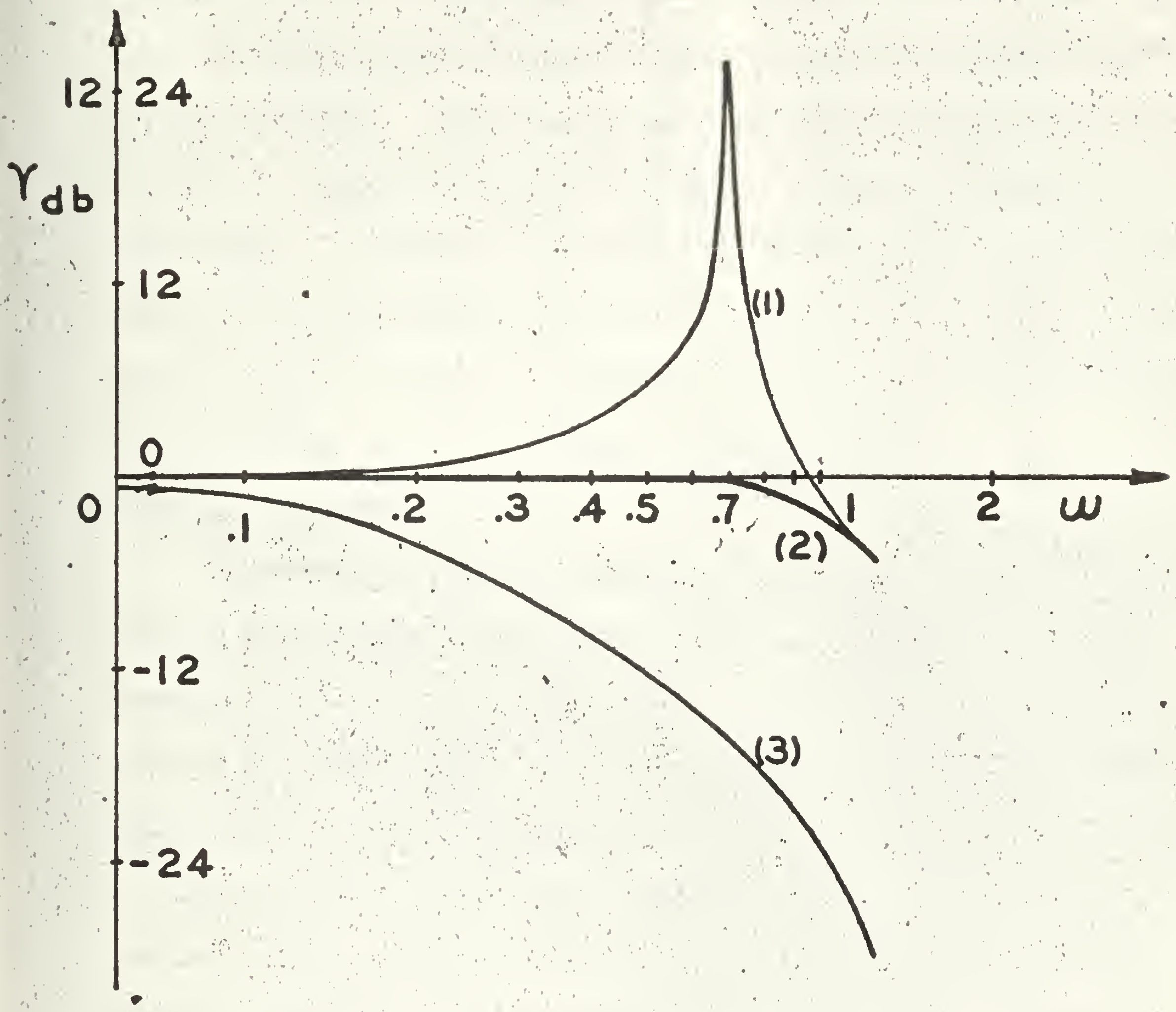


Fig. 2-7b Frequency response of a third order system (computer set up). (1) Underdamped, (2) Discontinuously damped, (3) Overdamped.

between 35 volts and 50 volts the system became underdamped, approaching curve 1, and for amplitudes from 7 to 0 volts the frequency response approached the overdamped condition of curve (3). In taking the data many output wave shapes were distorted, and the amplitude of the fundamental was estimated. The R-C differentiation was not used for these curves. Fig. 2-8 uses the phase plane to show the effect of the R-C differentiator on the system response. It is readily seen that the performance using the R-C circuit is negligibly different from that using the theoretically exact relationships. Early and late switching can be accomplished in this case also, by merely adjusting the coefficient of the E term in the switching equation. The results are essentially the same as for the second order case (See Fig. 2-5c) and the same comments apply.

Fourth Order System.

To extend the theory to higher order systems, to verify its applicability to actual physical systems, and to investigate the need for (and effect of) practical approximations to the mathematical requirements, a closed loop positioning servo was assembled. This consisted of a D. C. amplifier, amplidyne generator, and a 1/4 H.P. shunt motor in cascade. D. C. excited potentiometers were used for an error detector, the load was a large inertia, and a D. C. tachometer was attached. The gain was set to provide a stable but badly underdamped system. Open loop frequency response tests provided a transfer function

$$\frac{\Theta_c}{E}(s) = \frac{23774}{s(s+8)(s+11.55+j11.8)(s+11.55-j11.8)} \qquad (2\text{-}20)$$

From this the roots of the uncompensated system are at $-0.35 \mp j8$ and $-15.15 \mp j11.89$. During the tests RC differentiating filters were cascaded with the tachometer and various derivative signals observed. The second derivative signal was quite noisy and the third derivative signal

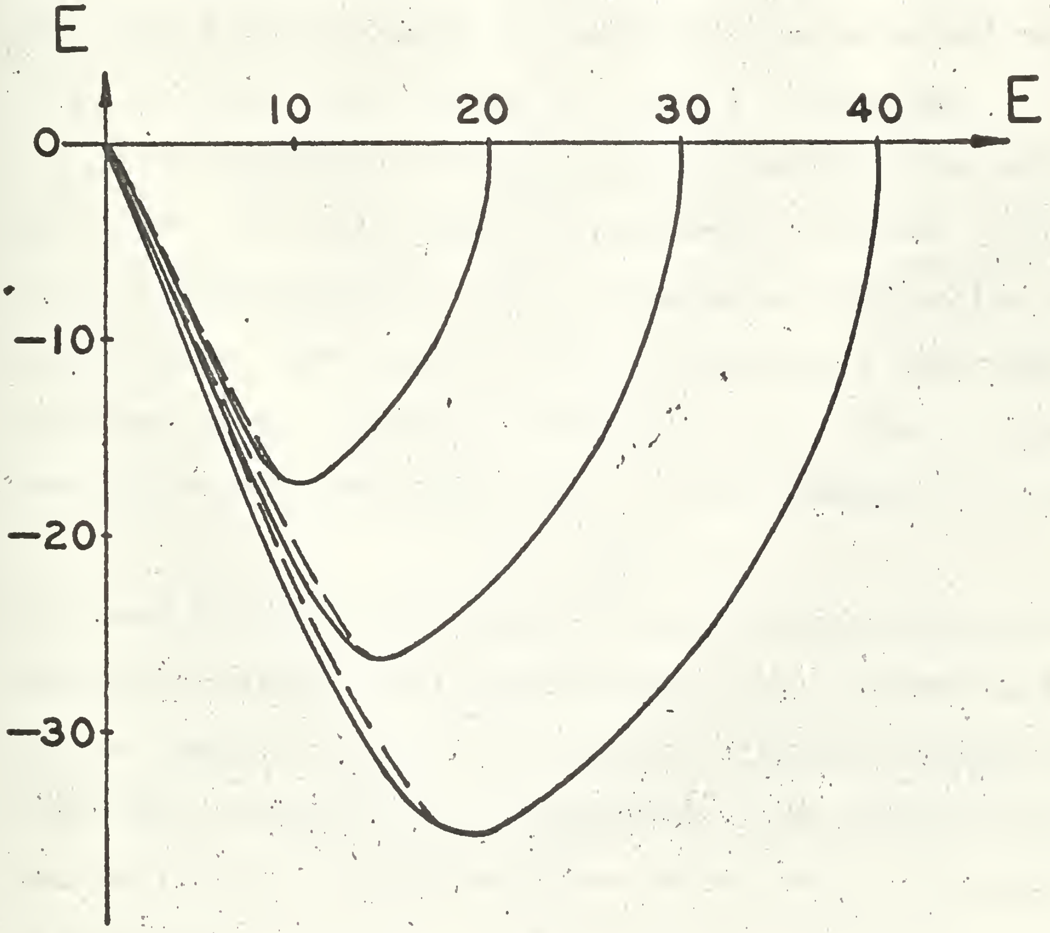


Fig. 2-8 Comparison of response using R-C differentiator instead of computer differentiator.

________________ Computer differentiator
\- - - - - - - - - - R-C differentiator

was too noisy to use. It was decided not to use the third derivative signal at all, and preferably not the second derivative signal. However, it is easily shown that the system cannot be overdamped using tachometer feedback only, but using both first and second derivative feedback roots can be placed at -3; -6; -11.5 $\mp$ j33.7. While complex conjugate roots exist the system is completely overdamped for a step displacement input. The second derivative signal was too noisy to use in the switching computer, thus the computer had to operate on E and $\dot{E}$ signals only.

It was considered desirable to study the system on the analog computer first. The under damped and overdamped systems were simulated with roots as indicated and the correct hyperplane switching surface computed using the same scheme applied to the second and third order cases. Typical transient response curves are shown on Fig. 2-9. A simplified switching equation was instrumented using only E and $\dot{E}$ signals in the form

$$E + A\dot{E} = 0 \tag{2-21}$$

The theory and calculation of coefficient A, and some additional instrumentation requirements have been discussed. Fig. 2-10 uses the E vs. $\dot{E}$ plane to compare the results of switching at the theoretically correct hyperplane with those obtained by switching at the hyperplane defined by equation (2-21). Fig. 2-11 indicates that the choice of the coefficient A is not critical.

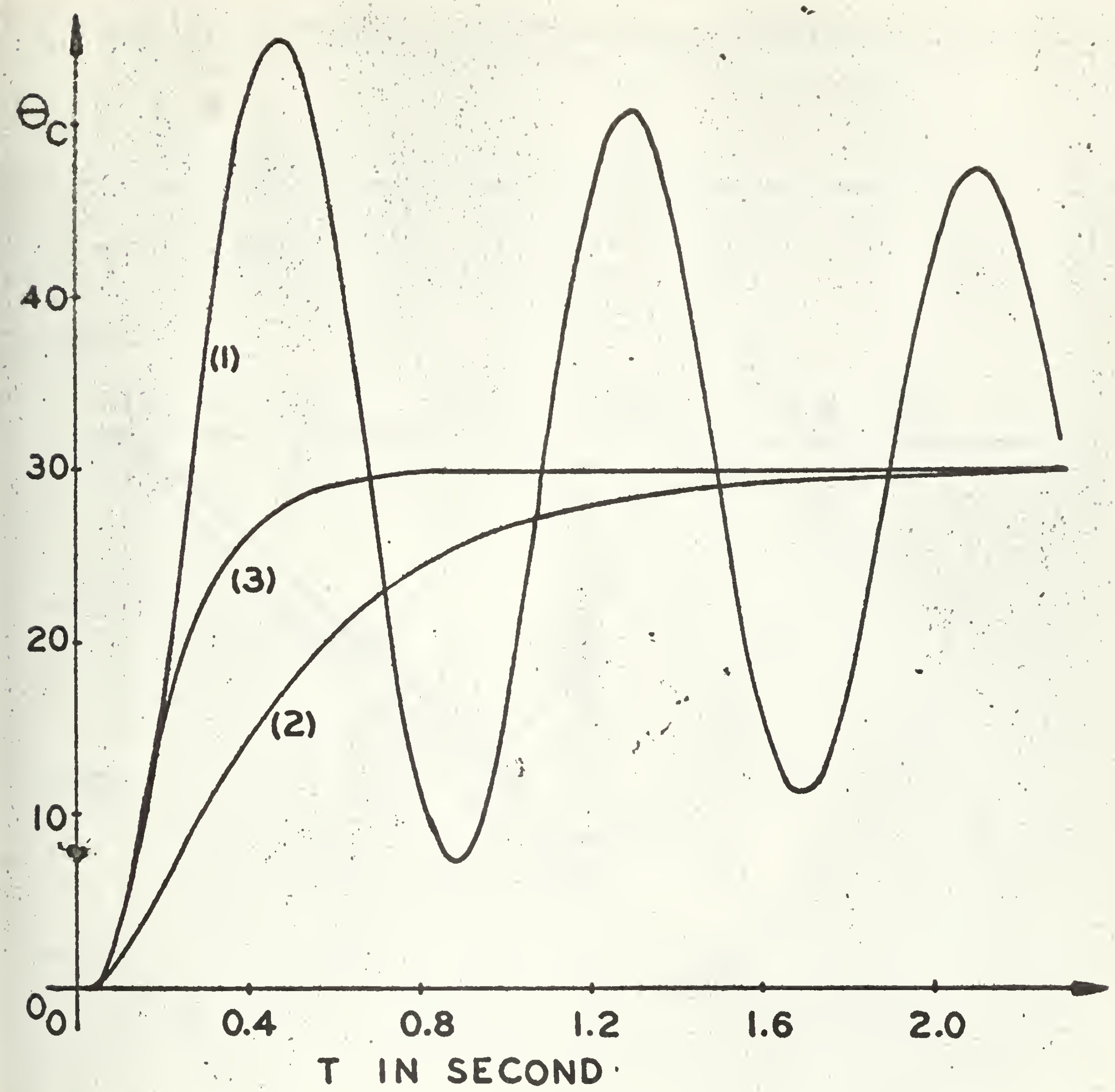


Fig. 2-9 Step response of a fourth order system.
(1) Underdamped, (2) Overdamped, (3) Discontinuously damped.

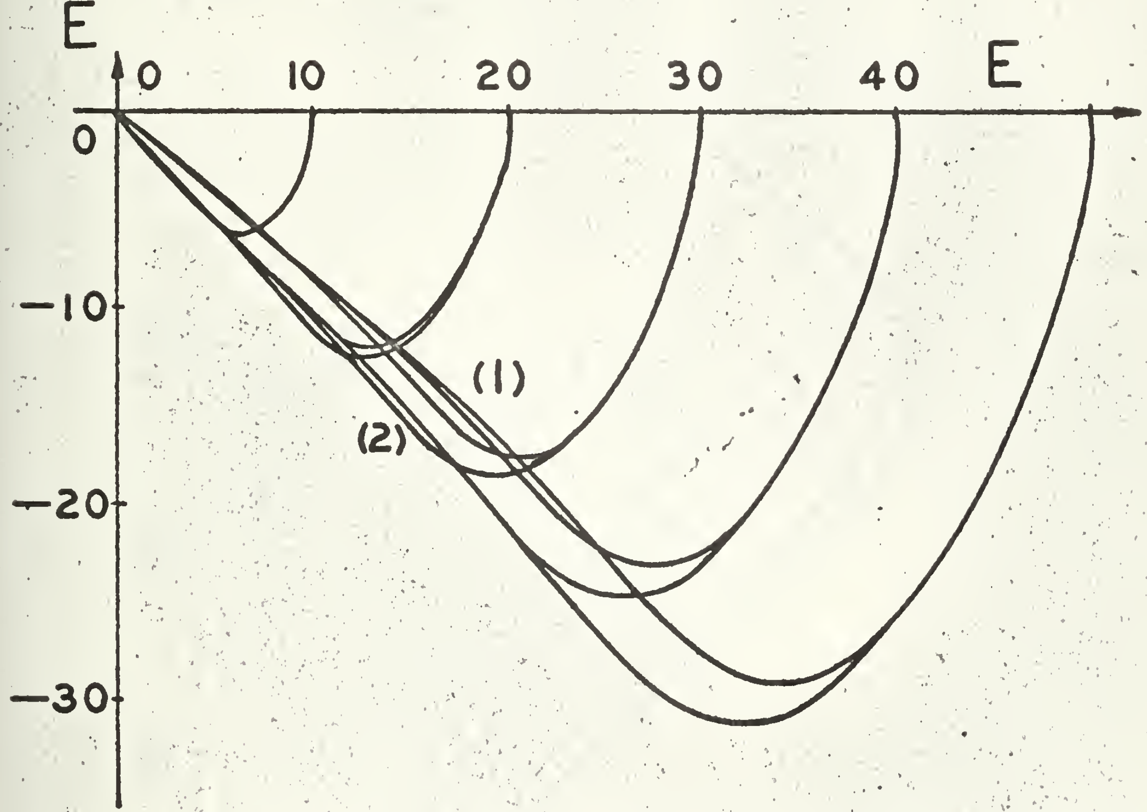


Fig. 2-10 Comparison of response. (1) Using hyperplane switching, (2) Using E vs. Ė switching.

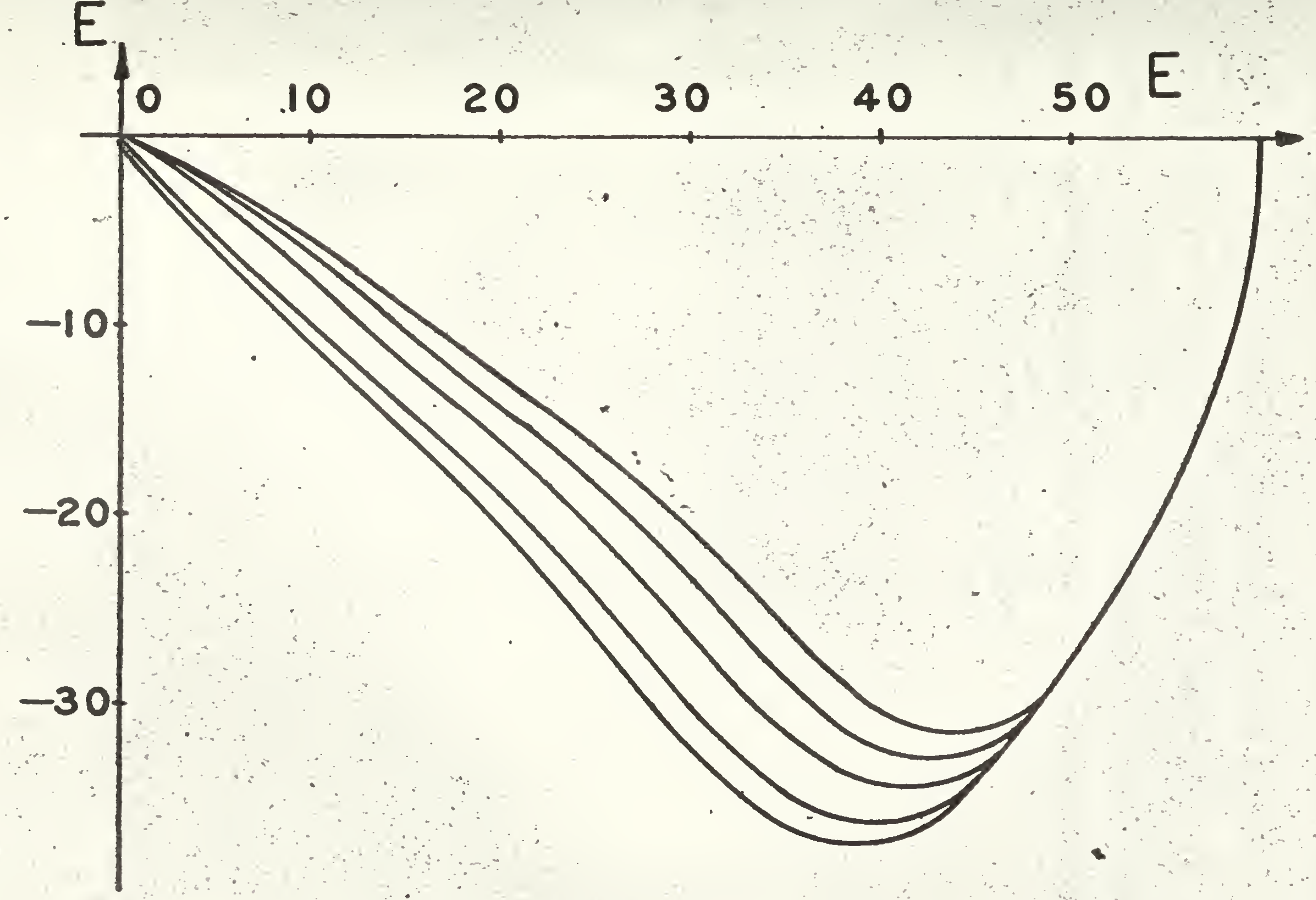


Fig. 2-11 Effect of switching time on response, computer study.

SECTION VI - PHYSICAL SYSTEM VERIFICATION

Fig. 2-12 shows the schematic diagram of the control system tested. Parameter values are indicated. The modifications of the switching circuit are necessary to prevent reopening of the compensation loop. Fig. 2-13a shows a family of transient response curves obtained from the physical system, with range of step amplitudes in excess of 100 degrees. Fig. 2-13b shows the effect of varying the coefficient A in the switching equation. It is apparent that acceptable step responses can be obtained over a reasonable range of adjustments of the switching computer. (For details see part two of this chapter).

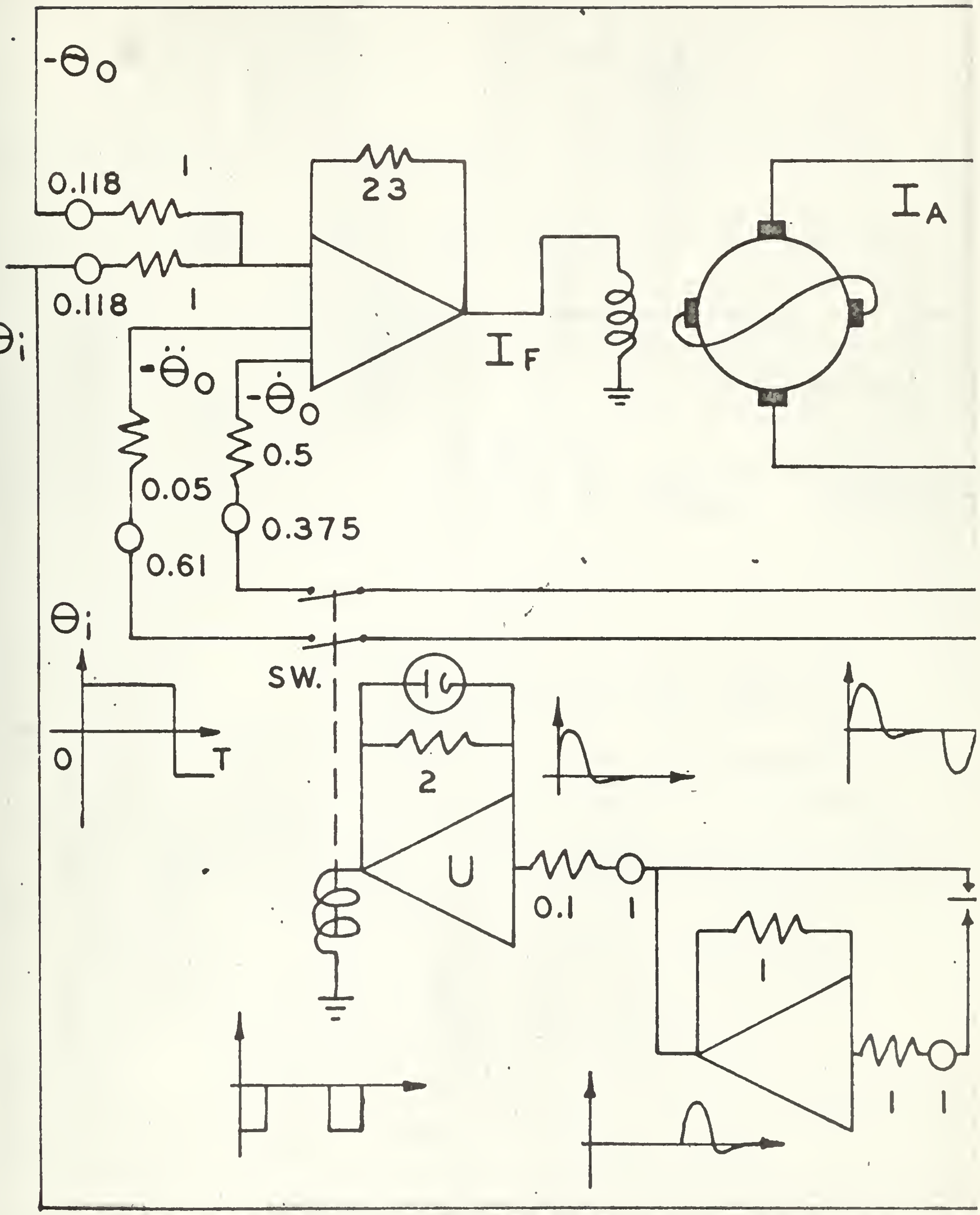
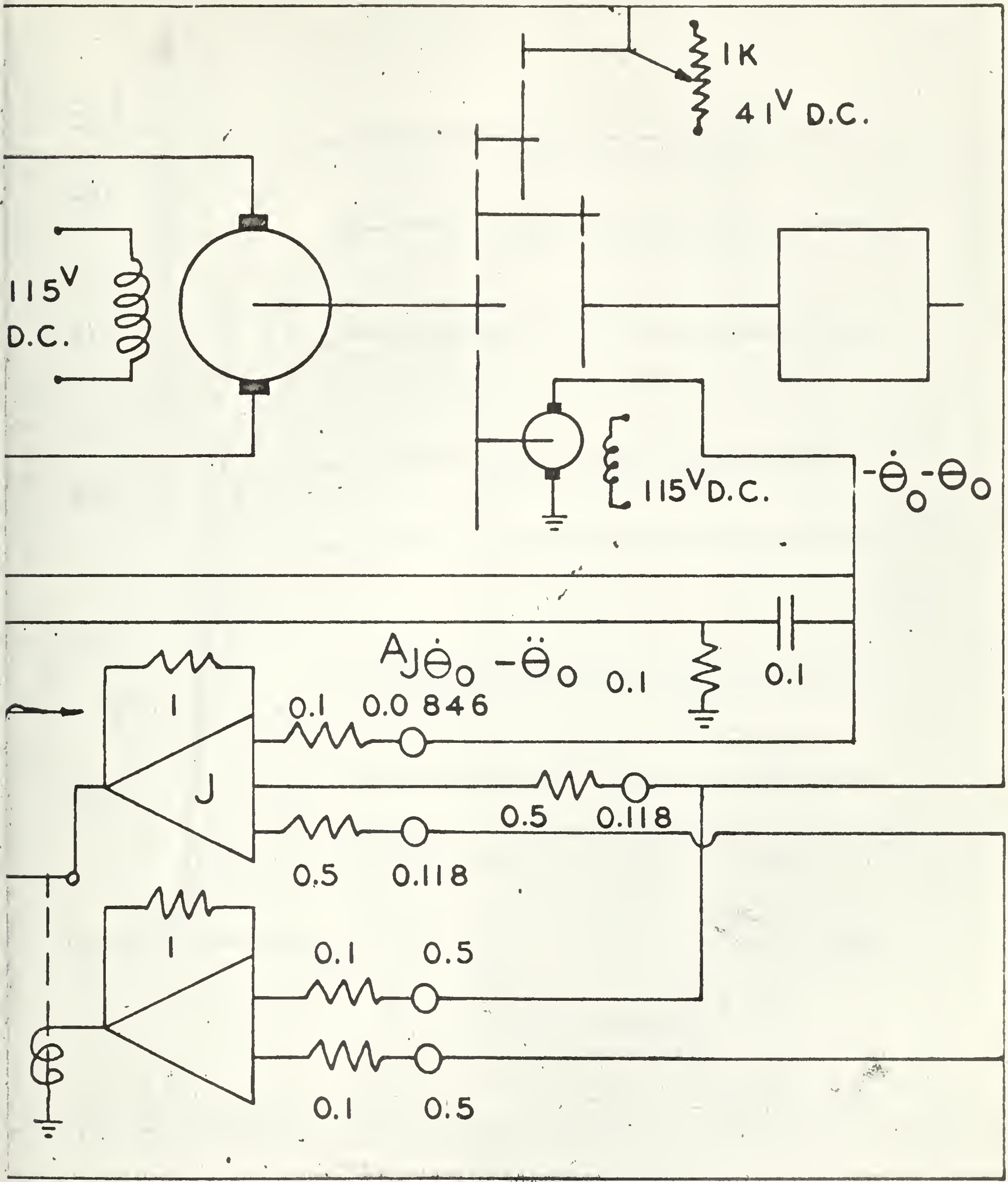


Fig. 2-12 Schematic Diagram of the Tested 4th Order System



1K
41V D.C.
115V
D.C.
115V D.C.
$-\dot{\theta}_0 - \theta_0$
$A_J\dot{\theta}_0$ $-\ddot{\theta}_0$
0.1
0.1
I
0.1 0.0846
J
0.5 0.118
0.5 0.118
I
0.1 0.5
0.1 0.5

Fig. 2-12

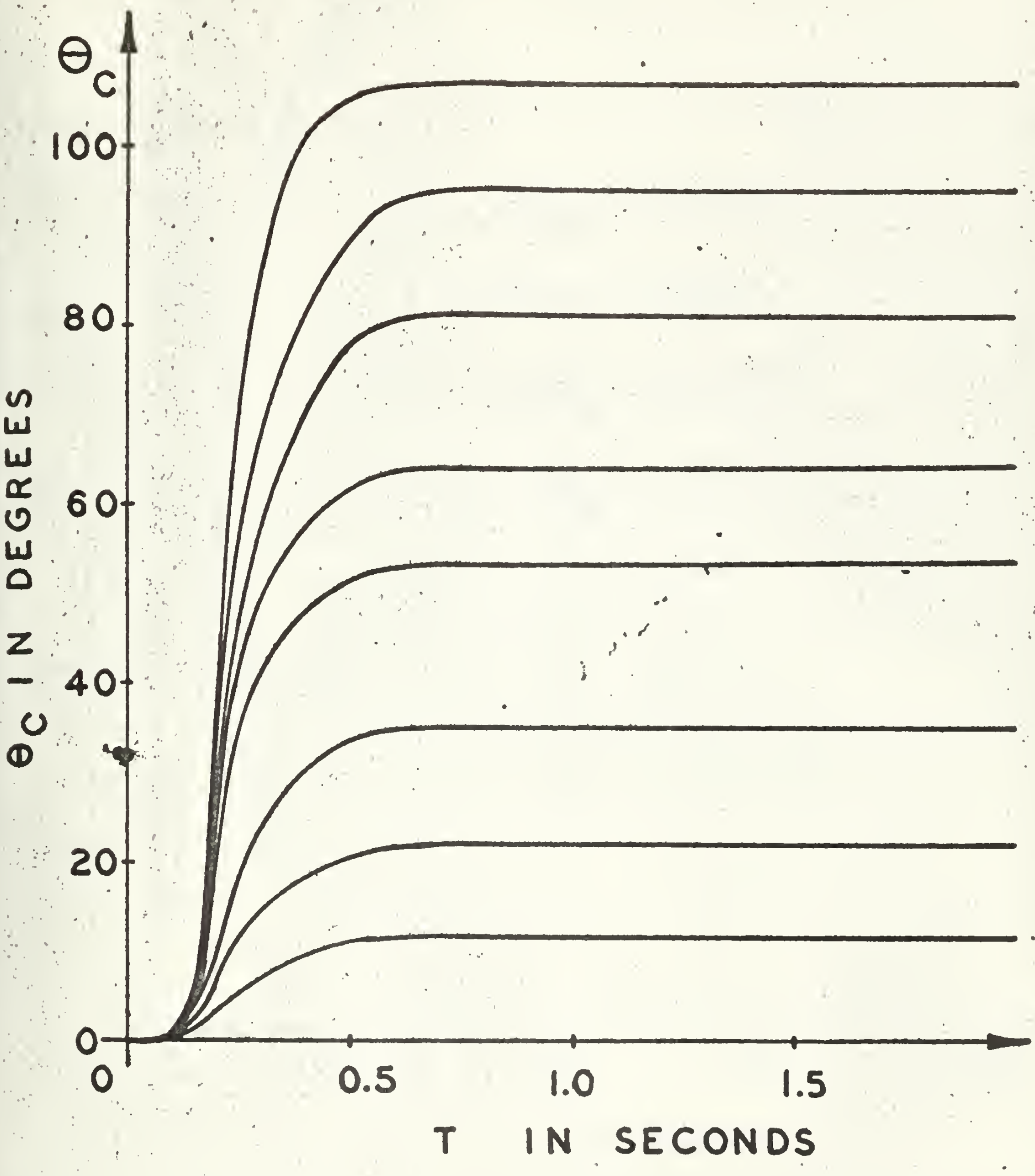


Fig. 2-13a Transient response for different magnitudes of step input.

IN DEGREES

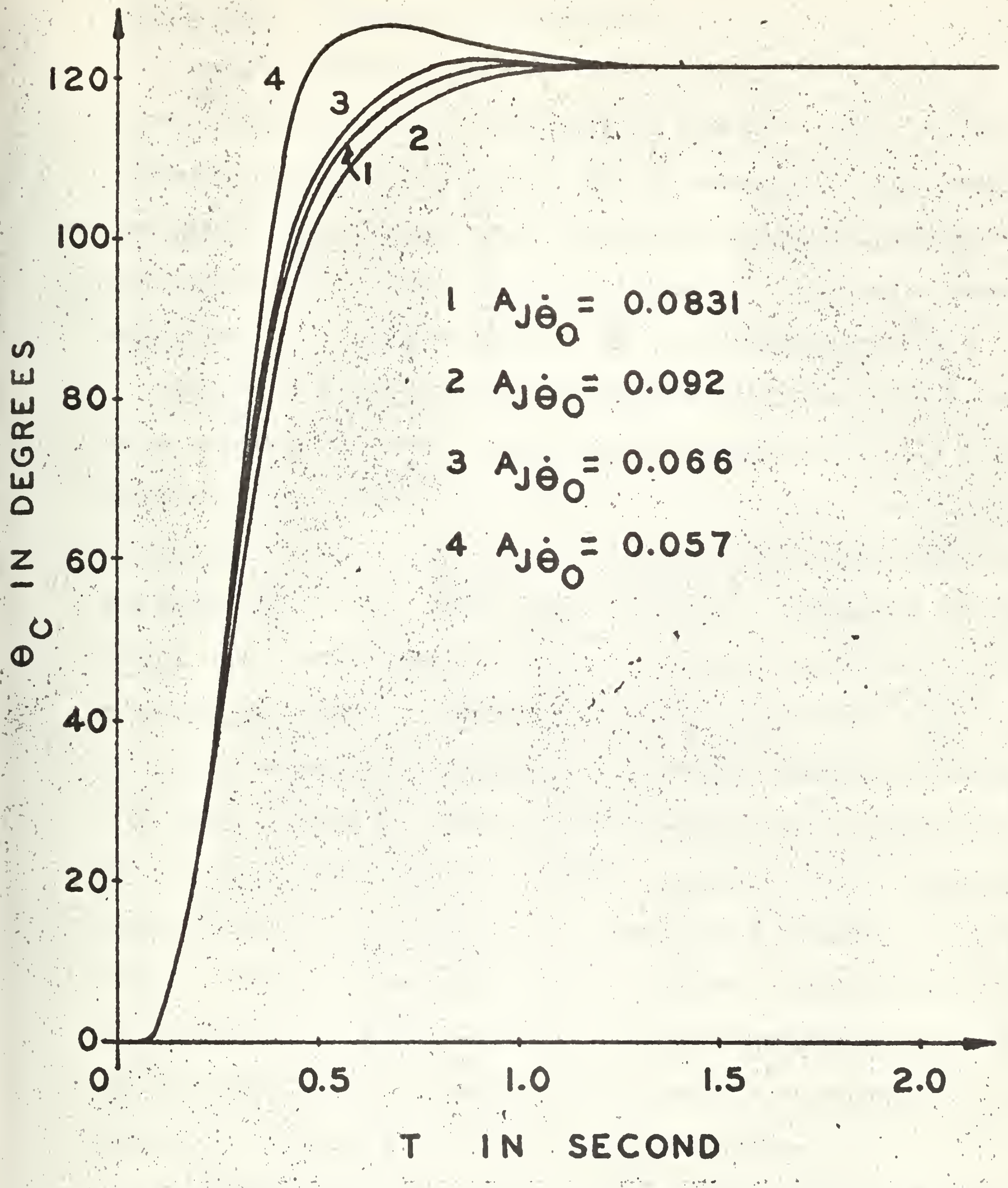


Fig. 2-13b Effects of switching time upon transient response. See Fig. 2-12 for $AJ\theta_o$.

SECTION VII - DISCUSSION AND CONCLUSIONS

From the equations, circuits and computer studies presented it is apparent that the concept of discontinuous damping of linear control systems is a relatively simple concept; the compensation loops required are easily determined, the parameter values easily calculated, and the switching computer for step inputs is readily designed. The results show a consistently fast, deadbeat response for step displacement inputs.

When the principles are applied to physical systems, no difficulties are anticipated with second and third order systems, since only first and second derivative signals are required. For higher order systems some difficulties are encountered due to the noise levels in the second, third, and higher derivatives. These difficulties may be avoided for the case of step displacement inputs by the simple procedure of not using the higher derivative signals. Feedback loops utilizing only lower derivatives can usually compensate the system so that its step response is overdamped. A switching computer may also be designed without using higher derivatives. For step displacement inputs a switching hyperplane giving precisely correct switching for any order system may be instrumented using only E and $\dot{E}$ signals. The theory has been verified by computer simulation and by design and test of an actual system. It has been shown that suitable engineering approximations are available for the generation of derivative signals and for the formulation of a practical switching computer.

It should be noted that this type of system is quite insensitive to disturbances. Because the compensation used is solely of the derivative feedback type no attenuation is introduced when the feedback circuits are closed. Thus under static conditions any load disturbance is not only opposed by a maximum forward gain but also by maximum damping. Thus oscillations are not likely to occur unless the load disturbance is

severe enough to make the relay open the damping circuits.

CHAPTER II Part II DETAIL CALCULATIONS, PLOTS, COMPUTER SETUP AND EXPERIMENT RESULTS

SECTION I- PHASE SPACE MODEL FOR INDICATING THE PLANES CORRESPONDING TO EIGENVECTORS

The equations used for Fig. 2-1 in Part I of this Chapter are

$$\dddot{\theta}_c + 2.02\,\ddot{\theta}_c + 0.54\,\dot{\theta}_c + \theta_c = \theta_R \qquad \text{(Under damped)} \quad (2\text{-}22)$$

$$\dddot{\theta}_c + 3.5\,\ddot{\theta}_c + 3.5\,\dot{\theta}_c + \theta_c = \theta_R \qquad \text{(Over damped)} \quad (2\text{-}23)$$

For step input the characteristic equations are

$$\dddot{E} + 2.02\,\ddot{E} + 0.54\,\dot{E} + E = 0 \qquad (2\text{-}24)$$

$$\dddot{E} + 3.5\,\ddot{E} + 3.5\,\dot{E} + E = 0 \qquad (2\text{-}25)$$

The real roots for Eq. (2-25) are at $-r_1, -r_2, -r_3$

where $r_1 = 0.5$, $r_2 = 1$, $r_3 = 2$

The equations for the three Hyperplanes are

$$u_1 = r_2 r_3 E + (r_2 + r_3)\dot{E} + \ddot{E} = 0 \qquad (2\text{-}26)$$

$$u_2 = r_1 r_3 E + (r_1 + r_3)\dot{E} + \ddot{E} = 0 \qquad (2\text{-}27)$$

$$u_3 = r_1 r_2 E + (r_1 + r_2)\dot{E} + \ddot{E} = 0 \qquad (2\text{-}28)$$

i.e.

$$u_1 = 2E + 3\dot{E} + \ddot{E} = 0 \qquad (2\text{-}29)$$

$$u_2 = E + 2.5\dot{E} + \ddot{E} = 0 \qquad (2\text{-}30)$$

$$u_3 = 0.5E + 1.5\dot{E} + \ddot{E} = 0 \qquad (2\text{-}31)$$

The computer setup is the same as in Fig. 2-6b. The setting values are in the table below (Table 2-1). The recorded plots are in Fig. 2-14a, b, c.

TABLE 2-1 Computer Setting Values for Phase Space Model for Fig. 2-1.

From the equations of Computer setup for Fig. 2-6b.

$W_4' = 0.583$ or $W_4' = 0.583 - 0.337 = 0.246$

$W_5' = 1.166$ $W_5' = 1.166 - 0.18 = 0.986$

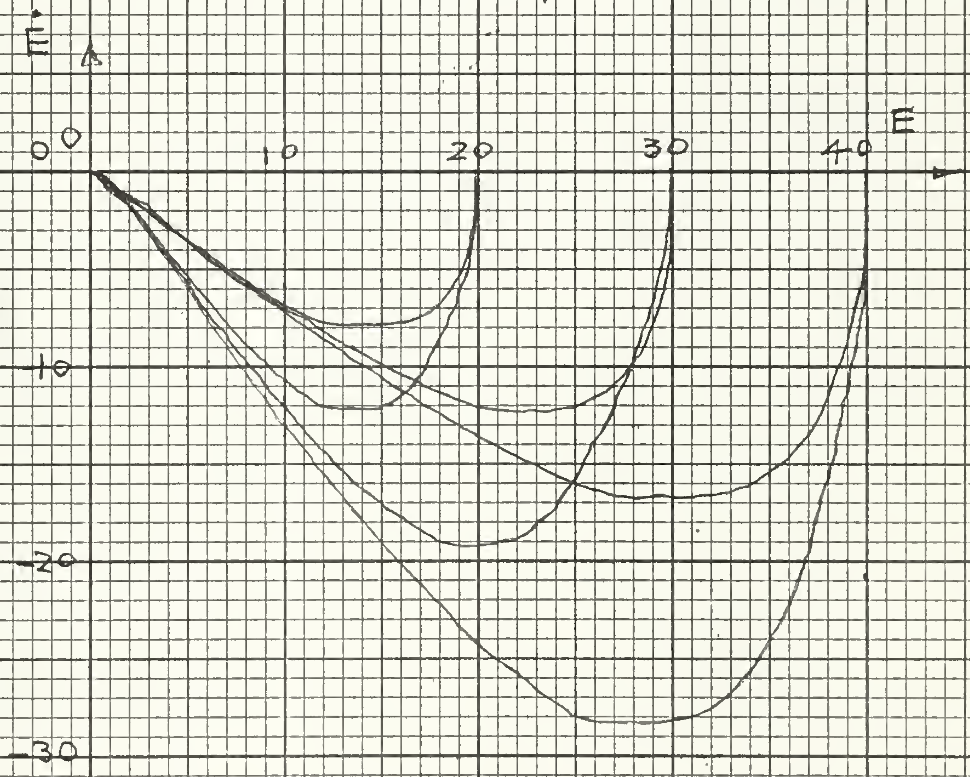
For $\alpha_u = 0.1$

$\left.\begin{array}{l} W_{11} = 0.6 \\ W_{12} = 0.6 \\ W_{13} = 0.1 \end{array}\right\}$ For μ_1

$\left.\begin{array}{l} W_{11} = 0.3 \\ W_{12} = 0.6 \\ W_{13} = 0.1 \end{array}\right\}$ For μ_2

$\left.\begin{array}{l} W_{11} = 0.15 \\ W_{12} = 0.3 \\ W_{13} = 0.1 \end{array}\right\}$ For μ_3

Fig. 2-14a Computer plots for making a phase space model for Fig. 2-1.



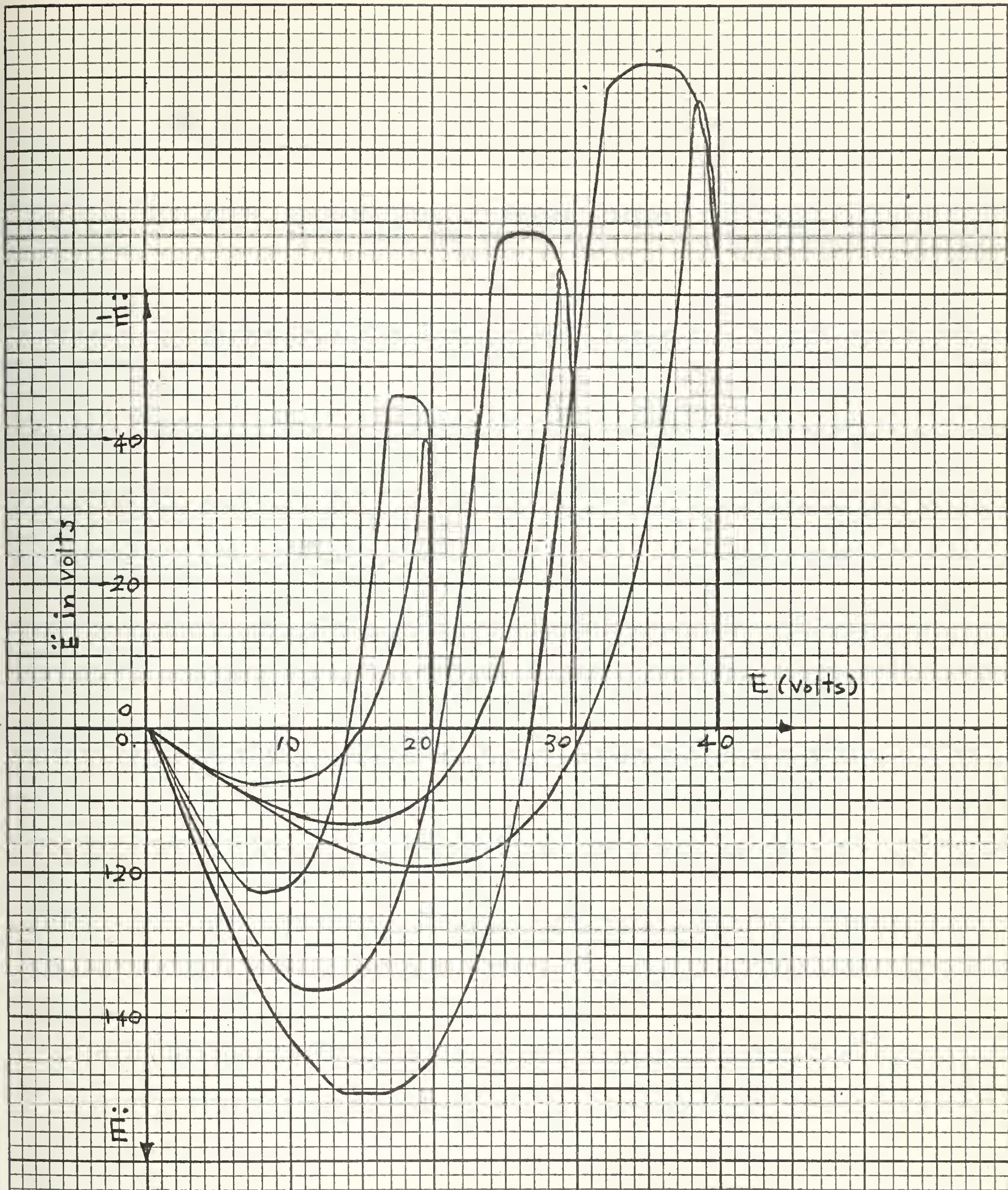
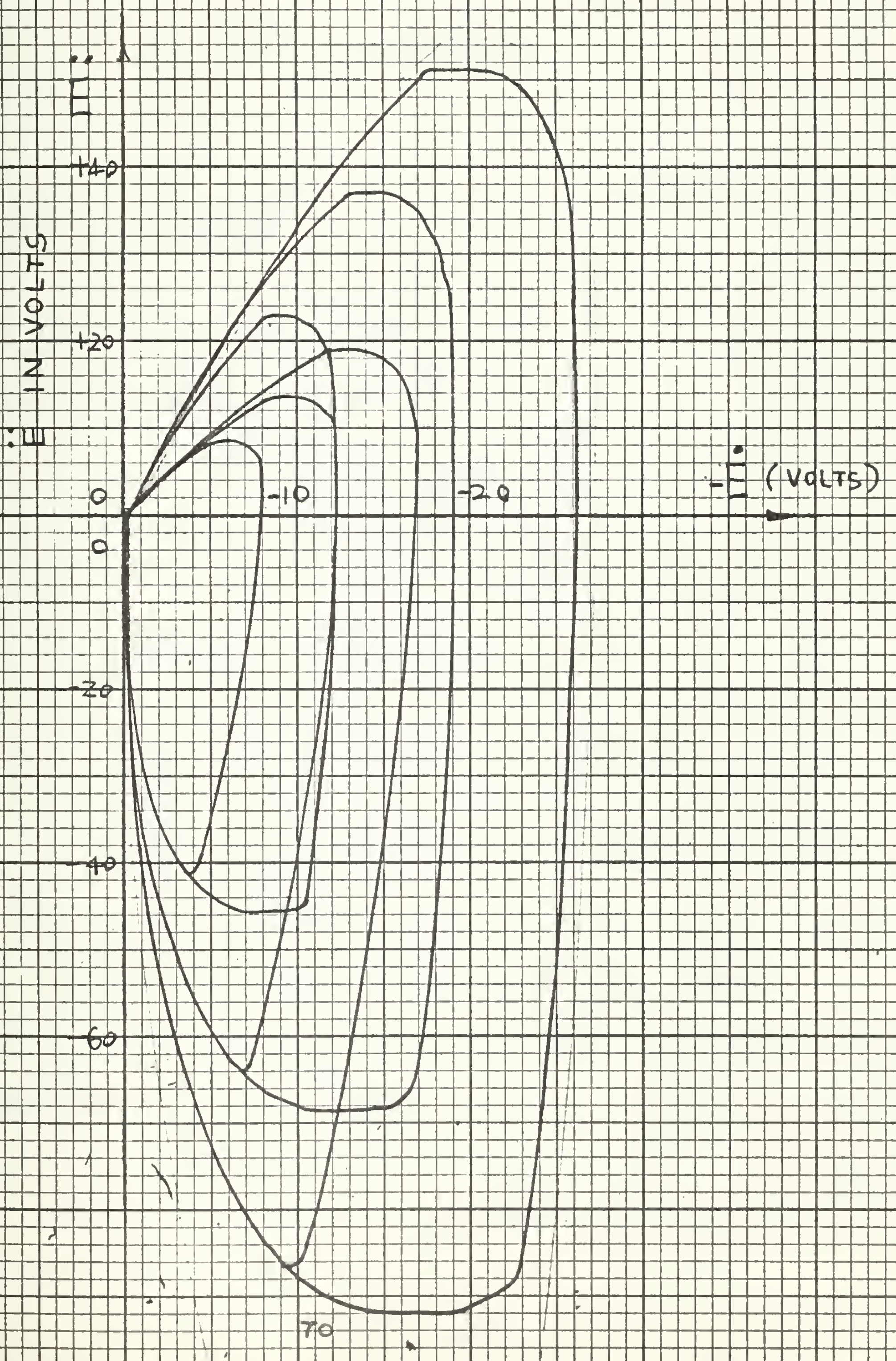


Fig. 2-14b Computer plot for making a phase space model for Fig. 2-1.

Fig. 2-14c Computer plot for making a phase space model for Fig. 2-1.



SECTION II - SECOND ORDER SYSTEMS

(A) Phase Plane Analysis of a Second Order Servo System with Discontinuous Tach. Feedback.

From Fig. 2-3a - For under damped case

$$F_o(S) = \frac{64}{S(S+2.6)} \tag{2-32}$$

$$F_c(s) = \frac{64}{s^2 + 2.6s + 64} \tag{2-33}$$

For a step input

$$\ddot{E} + 2.6\dot{E} + 64E = 0 \quad , \quad \zeta = 0.162 \; , \; \omega_n = 8 \tag{2-34}$$

Let $N = \dfrac{d(\dot{E}/\omega_n)}{dE}$

Then $\dfrac{\dot{E}}{\omega_n} = \dfrac{-E}{N + 0.352}$

For over damped case, h = 0.3

$$F_o(S) = \frac{64}{S^2 + 21.8S} \tag{2-35}$$

$$F_c(S) = \frac{64}{s^2 + 21.8S + 64} \tag{2-36}$$

Let $S^2 + 21.8S + 64 = 0$ then $r_1 = -18.304 \; , \; r_2 = -3.496$

and $\dfrac{\dot{E}}{\omega_n} = \dfrac{-E}{N + 2.725} = kE$

For $\dfrac{r_2}{\omega_n} = \dfrac{-3.496}{8} = -0.437 \; , \; N = k = -0.437$ (2-37)

$\dfrac{r_1}{\omega_n} = \dfrac{-18.304}{8} = -2.288 \; , \; N = k = -2.288$ (2-38)

Then when the switch is closed on these two lines, the trajectory will be following a straight line and goes to the origin.

If switching occurs near this line, the trajectory will still go to the origin with no overshoot or undershoot. But for large value of tachometer feedback the switching operation needs to be accurate, otherwise overshoot or undershoot will occur. The calculated values and plots are given in Table 2-2 and Fig. 2-15, 2-16.

TABLE 2-2 Data for Phase Plane Plot

$$N = \frac{d(\dot{E}/8)}{dE} \qquad K = \frac{\dot{E}/8}{E}$$

Under Damped System		Over Damped System h = 0.3	
N	K	N	K
0	-3.075	0	-0.367
0.1	-2.35	1	-0.269
0.2	-1.91	-1	-0.58
0.3	-1.6	-2	-1.38
0.475	-1.25	-2.2	-1.9
1.0	-0.755	-2.3	-2.35
1.345	-0.6	-2.4	-3.075
2.0	-0.42	-2.35	-2.66
4.0	-0.231		
8.0	-0.121	-0.437	-0.437
-0.1	-4.44	-2.288	-2.288
-0.2	-8		
-3.25	00		

For h = 0.96

N	K
0	-0.125
-0.124	-0.124
-6.005	-0.5
-7.005	-1
-7.5	-2
-7.84	-6
-7.8781	-7.8781
-7.9	-9.51

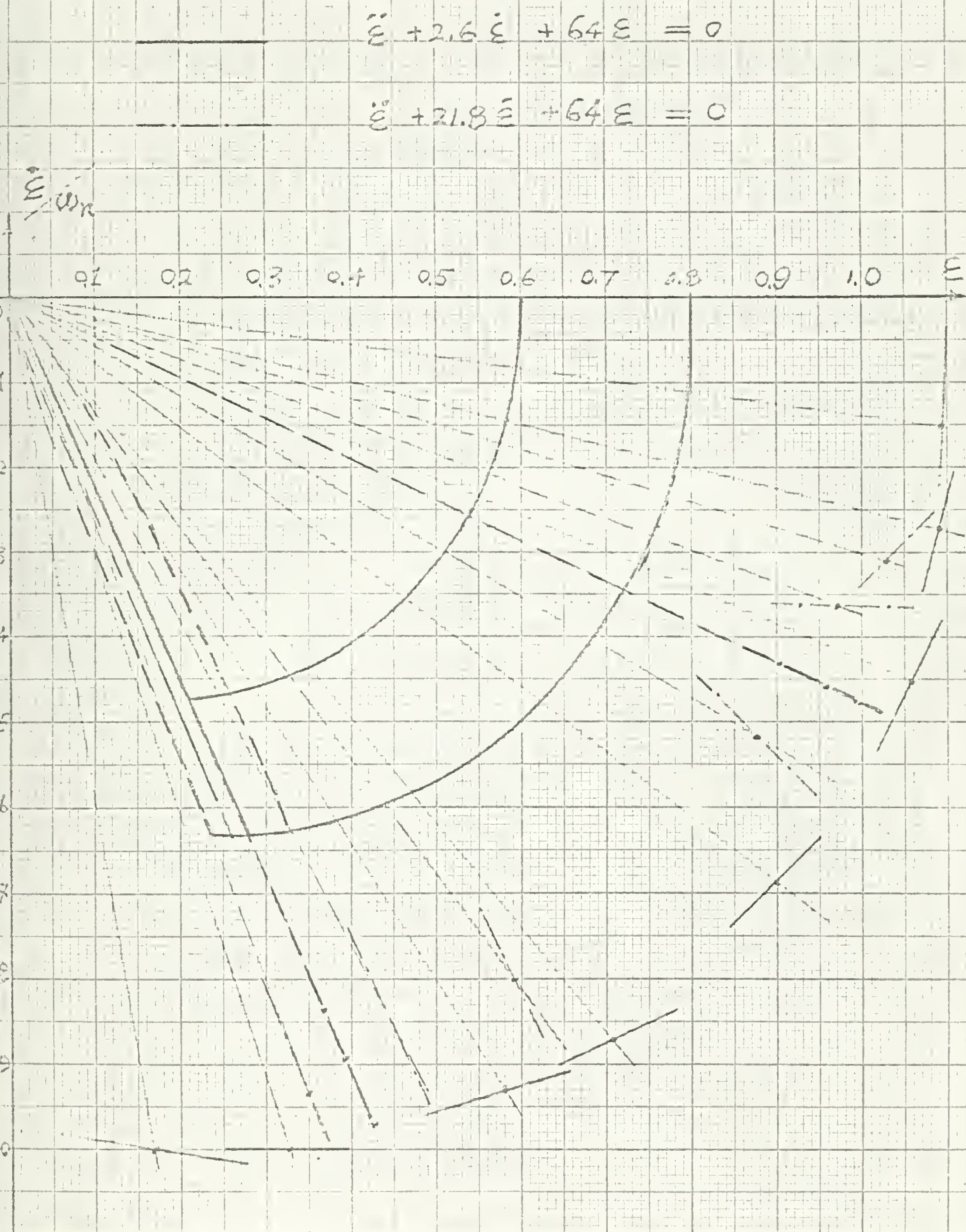


Fig. 2-15 Phase Plane Plot of a Second Order Servo with Discontinuous Tach. Feedback. (h = 0.3).

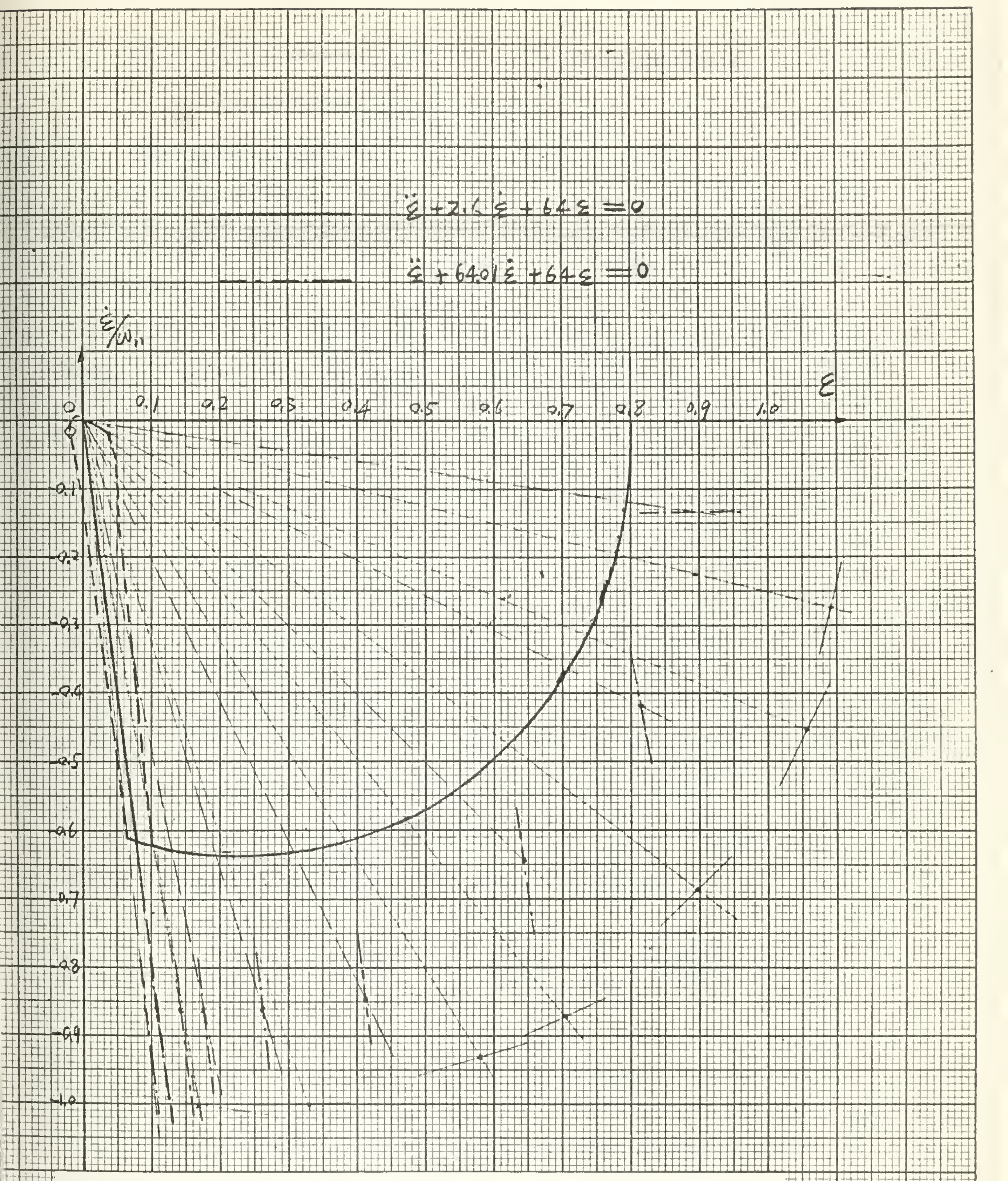


Fig. 2-16 Phase Plane Plot of a Second Order Servo with Large Discontinuous Tach. Feedback. (h = 0.96).

(B) Analog Computer Study of a Second Order Servosystem with Discontinuous Tachometer Feedback

Mathematical Solution

From Fig. 2-3a, for h = o

$$\overline{\theta}_c = \frac{64}{S^2+2.6S+64}\ \overline{\theta}_R = \frac{\omega_n^2}{S^2+2\rho\omega_n S+\omega_n^2}\ \overline{\theta}_R \tag{2-39}$$

For unit step input

$$\theta_c = \left[1 - \frac{\omega_n}{\omega_c} e^{-\alpha t} \sin(\omega_c t + \Upsilon)\right]\mu(t) \tag{2-40}$$

where $\Upsilon = 80.7°$, $\alpha = \rho\omega_n$, $\omega_c = \omega_n\sqrt{1-\rho^2}$

For $\theta_i = 0.5\mu(t)$, $\dot{\theta}_c = \frac{0.5\,\omega_n^2}{\omega_c} e^{-\alpha t}\sin(\omega_c t)\,\mu(t)$ (2-41)

$\theta_{c\,max.} = 0.797$ at $t = 0.398$ Sec.

$\dot{\theta}_{c\,max.} = 3.16$ at $t = 0.1785$ Sec.

$J_{max.} = 16$

For h = 0.3

$$F_c(s) = \frac{64}{S^2+21.8S+64} \tag{2-42}$$

poles at -18.304 and -3.496. The equation for the switching line is

$$u = 18.304E - \dot{E} = 0 \tag{2-43}$$

For step input $u = 18.304E - \dot{\theta}_c = 0$ (2-44)

Computer setup for Fig. 2-4 in Part I.

$$[-J] = \left[\frac{32\alpha_{\theta i}}{\alpha_J}\overline{\theta}_i - \frac{32\alpha_{\theta o}}{\alpha_J}\overline{\theta}_o - \frac{32h\alpha_{\omega o}}{\alpha_J}\overline{\omega}_o\right]$$
$$= W_1\overline{\theta}_i - W_2\overline{\theta}_o - W_h\overline{\dot{\theta}}_o \tag{2-45}$$

$$\overline{\omega}_o = \frac{\frac{f}{J}J}{S+f/J} = -\frac{W_3}{P+W_4}[-J] \tag{2-46}$$

$$W_3 = \frac{\alpha_J f/J}{\alpha_{\omega o}\alpha_t},\quad W_4 = \frac{f/J}{\alpha_t}$$

$$[-\overline{\theta}_o] = -\frac{1}{P}(W_5\,\overline{\omega}_o) \tag{2-47}$$

$$W_5 = \frac{\alpha_{\omega o}}{\alpha_{\theta o}\alpha_t}$$

$$\overline{\mu} = W_7\theta_i - W_8\theta_o - W_9\dot{\theta}_o \tag{2-48}$$

For $\alpha_t = 20$

$\alpha_x = \alpha_u = \alpha_{\theta i} = \alpha_{\theta o} = 0.00996$

$\alpha_{\omega i} = 0.0704$

$\alpha_J = 0.32$

Then

$W_1 = 0.996$	$Rf_1 = 10$	$R_1 = 1$	$a_1 = 0.0996$
$W_2 = 0.996$	"	$R_2 = 1$	$a_2 = 0.0996$
$W_h = 2.11$	"	$R_h = 1$	$a_h = 0.211$
$W_3 = 0.455$	$Cf_1 = 1.06$	$R_3 = 1$	$a_3 = 0.482$
$W_4 = 0.13$	"	$R_4 = 1$	$a_4 = 0.138$
$W_5 = 0.354$	$Cf_2 = 1.06$	$R_5 = 1$	$a_5 = 0.375$
$W_6 = 1$	$Rf_2 = 1$	$R_6 = 0.5$	$a_6 = 0.5$
$W_7 = W_8 = 1$	$Rf_u = 1$	$R_7 = R_8 = 0.5$	$a_7 = a_8 = 0.5$
$W_9 = 0.385$	"	$R_9 = 1$	$a_9 = 0.385$

The setting values for control signals are W_7, W_8 and W_9 in the above table.

Note: The u signal in this set is only for step input.
For other inputs or operated with initial conditions not equal to zero the u signal must be produced by E & $\dot{E}$ instead of E & $-\ddot{\theta}_o$

SECTION III - THIRD ORDER SYSTEMS

(A) Mathematical Analyses of a Third Order Servo System with Discontinuous damping

From Fig. 2-6a

$$F_c(S) = \frac{K}{S(S+P_1)(S+P_2)} = \frac{0.000728}{S(S+0.02852)(S-0.1527)} = \frac{N_o}{D_o} \quad (2\text{-}49)$$

$$F_c(S) = \frac{F_o}{1+F_o(m^2S^2+hS)} = \frac{N_o}{D_o+N_o(m^2S^2+hS)} \quad (2\text{-}50)$$

$$= \frac{0.000728}{S^3+0.18122S^2+(0.0285\times0.1527)S+0.000728(m^2S+hS)}$$

(1) Underdamped case:

$$F_c(S) = \frac{0.000728}{S^3+0.1812S^2+0.0285\times0.1527S+0.000728} \quad (2\text{-}51)$$

For step input $\theta_i = 1\mu(t)$

$$\theta_o(S) = \frac{0.000728}{S(S+0.180)(S+0.0006+j0.0635)(S+0.0006-j0.0635)} \quad (2\text{-}52)$$

where all the poles of $F_c(S)$ are found by root locus method in Fig. 2-17.

$$r_1 = -0.0006+j0.0635$$

$$r_2 = -0.0006-j0.0635$$

$$r_3 = -0.180$$

For $\theta_i = 1\mu(t)$ by using inverse Laplace Transformation the equation of θ_o is

$$\theta_o = 1 - 0.94765e^{-0.0006t}\sin(0.0635t+69.59°) - 0.118e^{-0.18t} \quad (2\text{-}53)$$

and $\dot{\theta}_o$ is

$$\dot{\theta}_o = 0.0005685e^{-0.0006t}\sin(0.0635t+69.59°)$$

$$-0.06175e^{-0.0006t}\cos(0.0635t+69.59°)+0.02124e^{-0.18t} \quad (2\text{-}54)$$

By differentiation of $\dot{\theta}_o$ the equation of $\ddot{\theta}_o$ is

$$\ddot{\theta}_o = 0.039211e^{-0.0006t}\sin(0.0635t+69.59°)+0.003832e^{-0.18t} \quad (2\text{-}55)$$

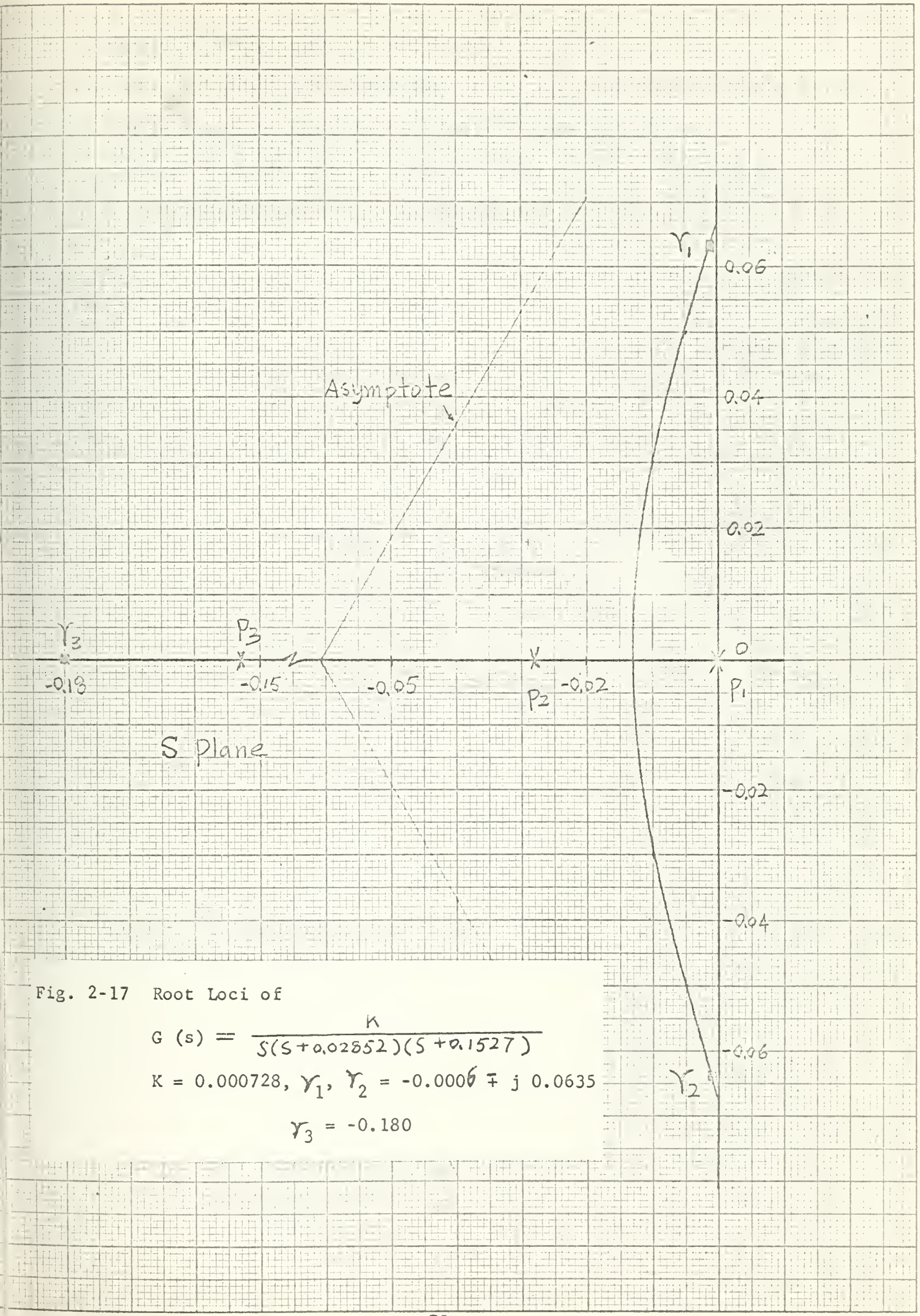


Fig. 2-17 Root Loci of

$$G(s) = \frac{K}{S(S+0.02852)(S+0.1527)}$$

$K = 0.000728$, $\gamma_1, \gamma_2 = -0.0006 \mp j\,0.0635$

$\gamma_3 = -0.180$

Here the small terms are neglected.

After introducing a time factor $\tau = 0.09t$, and substituting t in to these equations the result is Table 2-3 and Fig. 2-18.

where $x = \theta_i - \theta_o = 1 - \theta_o$, $\dot{x} = -\dot{\theta}_o$, $\ddot{x} = -\ddot{\theta}_o$

τ	0	1	1.97	3	4	5	6
x	-1	-0.8892	-0.46	+0.179	+0.716	+0.8955	+0.6637
$\dot{x}$	0	+0.2745	+0.57	+0.6364	+0.3852		
$\ddot{x}$	0	+0.370	+0.21	-0.1034	-0.3661		

Table 2-3 Transient response for a Negative Step Input.

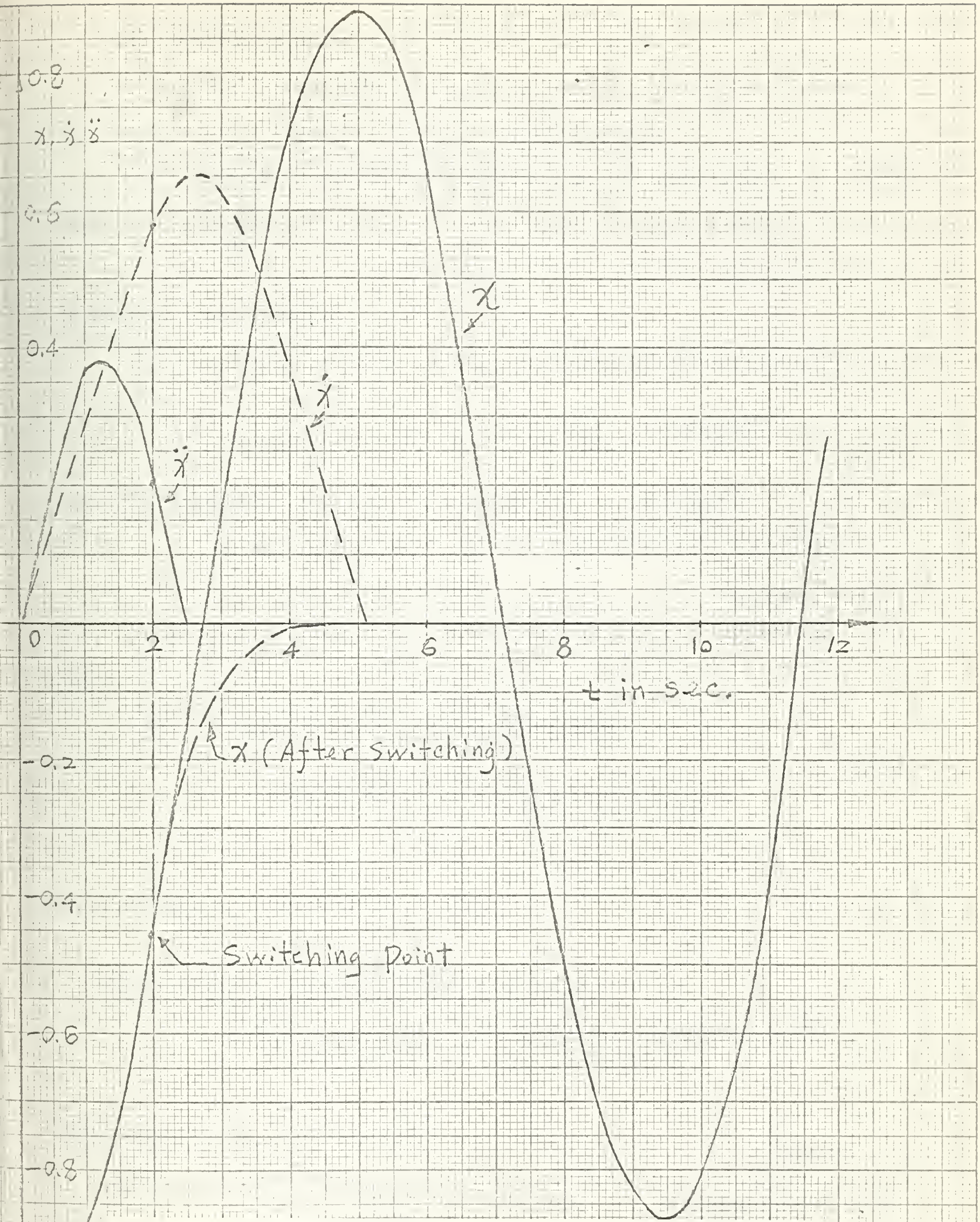


Fig. 2-18 Transient Response for a Negative Unit Step Input

(B) Overdamped Case: -

By choosing 3 real roots of $D_{F_c}(S)$ at -0.0115, -0.178 and -0.3539 the characteristic equation for the overdamped case is

$$\left[D^3 + 0.5439\,D^2 + 0.069\,D + 0.000728\right]\theta_o = 0.000728\,\theta_i \qquad (2\text{-}56)$$

where $D^n = \dfrac{d^n}{dt^n}$

$$\begin{bmatrix} h = 44.5 \\ m = 15.8 \end{bmatrix}$$

By Laplace Transformation

$$\Big\{ S^3 - S^2F(-0) - SF'(-0) - F''(-0) + 0.5439\left[S^2 - SF(-0) - F'(-0)\right]$$

$$+0.069\left[S - F(-0)\right] + 0.000728 \Big\}\bar{\theta}_o = \frac{0.000728}{S} \qquad (2\text{-}57)$$

i.e. $$\left[S^3 + 0.5439\,S^2 + 0.069\,S + 0.000728\right]\bar{\theta}_o = \frac{0.000728}{S}$$
$$+\Big[\theta_o(-0)S^2 - \dot{\theta}_o(-0)S - \ddot{\theta}_o(-0) - 0.543\,\theta_o(-0)\,S - 0.543\,\dot{\theta}_o(-0)$$
$$-0.069\,\theta_o(-0)\Big]$$

For $\theta_o(-0) = 0.5228$, $\dot{\theta}_o(-0) = 0.05368$, $\ddot{\theta}_o(-0) = 0.00189$

then

$$\bar{\theta}_o = \frac{0.000728}{S(S^3 + 0.543S^2 + 0.069S + 0.000728)} - \frac{0.5228S^2 + 0.33756S + 0.06711}{S^3 + 0.543S^2 + 0.069S + 0.000728} \qquad (2\text{-}58)$$

By using root locus methods in Fig. 2-19.

$$\bar{\theta}_o = \frac{0.000728}{S(S+0.0115)(S+0.178)(S+0.3539)} - \frac{0.5228(S+0.32326+j0.1548)(S+0.32326-j0.1548)}{(S+0.0115)(S+0.178)(S+0.3539)}$$

By inverse Laplace Transformation

$$\theta_o(t') = 0.000728\Big\{\frac{1}{0.0115\times0.178\times0.3539} - \frac{e^{-0.0115t'}}{0.1665\times0.3424\times0.0115}$$
$$+\frac{e^{-0.178t'}}{0.178\times0.1665\times0.1759} - \frac{e^{-0.3539t'}}{0.3539\times0.1759\times0.3424}\Big\}$$

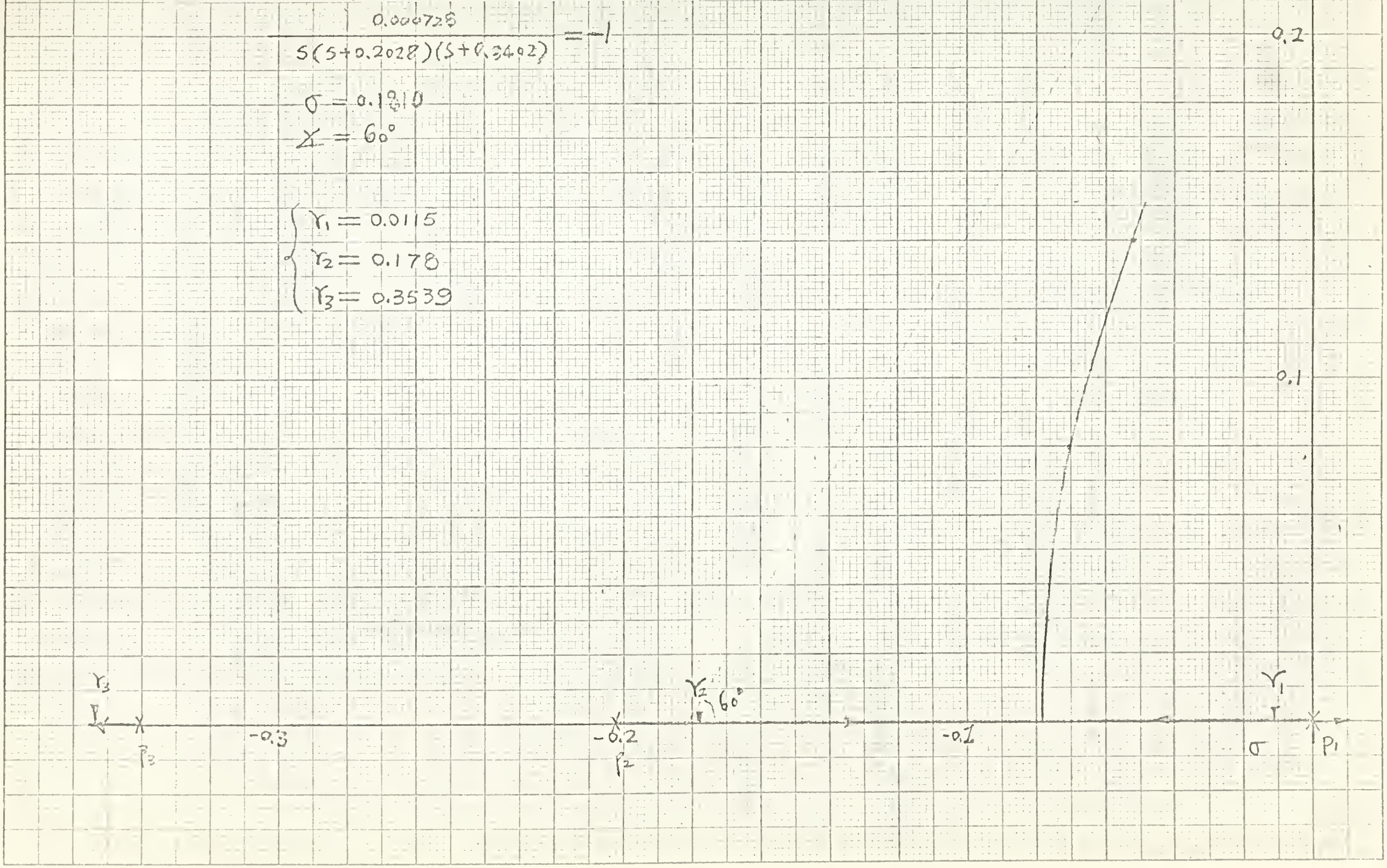
Fig. 2-19 Root Loci for Solving the Equation

$$\frac{0.000728}{S(S+0.2028)(S+0.3402)} = -1$$

$\sigma = 0.1810$

$\measuredangle = 60^\circ$

$$\begin{cases} r_1 = 0.0115 \\ r_2 = 0.178 \\ r_3 = 0.3539 \end{cases}$$



$$+0.5228\left[\frac{(0.348)^2 e^{-0.0115t'}}{0.1665\times0.3424}-\frac{(0.213)^2 e^{-0.178t'}}{0.1665\times0.1759}\right.$$

$$\left.-\frac{(0.16)^2 e^{-0.3539t'}}{0.3424\times0.1759}\right]$$

$$\theta_c(t') = 0.000728\,[1380.395-1525\,e^{-0.0115t'}+191.824\,e^{-0.178t'}$$

$$-46.9175\,e^{-0.3539t'}]+0.5228\,[2.124\,e^{-0.0115t'}$$

$$-1.5488\,e^{-0.178t'}+0.425\,e^{-0.3579t'}]$$

$$=1.404927-1.1102\,e^{-0.0115t'}+0.1396\,e^{-0.178'}-0.0342\,e^{-0.3539t'}$$

$$+1.1102\,e^{-0.0115t'}-0.8097\,e^{-0.178t'}+0.222\,e^{-0.3539t'}$$

$$=1+0.67\,e^{-0.178t'}+0.188\,e^{-0.3539t'} \tag{2-59}$$

Here in this $\theta_c(t')$ equation t' starts from $\tau = 1.97$ Sec,, or $t = 21.9$

t'	τ	t	θ_c
0	1.97	21.9	0.5118
10	2.873	31.9	0.8929
20	3.774	41.9	0.9866
30	4.675	51.9	0.9968

These calculations show that (1) at the switching time the coefficient of the most heavily damped term is cancelled by the initial condition at this instant. (2) This system has no over shoot.

Discussion About Time Domain Analysis of High Order System with Discontinuous Damping

From these detail calculations by using the Laplace transformation the various trajectories versus time can be calculated, and a phase space model can be constructed. The initial condition of each derivative signal can be clearly defined by plotting a straight line perpendicular to the time axis. Also the switching points in space can be found by plotting other curves according to the equations of hyperplanes.

In the die-out part (overdamped) of the trajectory, the effect of various initial conditions can be studied by putting initial conditions into the Laplace transform equations.

In higher order systems the same method can be used. By plotting all the derivative signals versus one time axis, it is easier to get a general view of the trajectory in higher order space, because a space model cannot be used for fourth or higher order systems. On the other hand, there is no differentiation problem as that in analog computer study. Therefore, the Laplace transformation method is a powerful tool for linear discontinuous damping analysis.

Analog Computer Study for a Third Order Servo System with Discontinuous Damping.

For the system in Section 2.1, after introducing a time scaling equation $\tau = 0.09t$:

$$\theta_c + 2.02\theta_o + 0.54\,\theta_o + \theta_c = \theta_i \qquad \text{underdamped (2-60)}$$

$$\theta_o + 6\,\theta_c + 8.5\,\theta_c + \theta_o = \theta_i \qquad \text{overdamped (2-61)}$$

$$\gamma_1 = 0.13 \qquad \gamma_2 = 2.0 \qquad \gamma_3 = 3.87$$

$$u = \gamma_2\gamma_3 x + (\gamma_2+\gamma_3)\dot{x} + \ddot{x} = 7.74x - 5.87\dot{\theta}_o - \ddot{\theta}_o \qquad (2\text{-}62)$$

By using the following magnitude scaling:

$$\alpha_\tau = 1$$

$$\alpha_{\theta i} = \alpha_{\theta o} = \alpha_x = \frac{1}{33.3} = 0.03$$

$$\alpha_{\dot\theta o} = \frac{0.667}{33.35} = 0.02$$

$$= \frac{0.38}{38} = 0.01$$

$$= \frac{7.74}{38.7} = 0.20$$

$$= \frac{0.2}{34.3} = 0.06$$

The equations for computer setup are:

$$\bar{x} = -\left\{\frac{\alpha_{\theta i}}{\alpha_x}\bar{\theta}_i - \frac{\alpha_{\theta o}}{\alpha_x}\bar{\theta}_o\right] = -[w_1\bar{\theta}_i - w_2\bar{\theta}_o] \qquad (2\text{-}63)$$

$$[-\bar{\dddot\theta}_o] = \frac{\alpha_{\theta i}}{\alpha_{\dddot\theta o}}(\theta_i - \theta_o) - 2.02\frac{\alpha_{\ddot\theta o}}{\alpha_{\dddot\theta o}}\bar{\ddot\theta}_o + 0.54\frac{\alpha_{\dot\theta o}}{\alpha_{\dddot\theta o}}\bar{\dot\theta}_o = w_3(\bar\theta_i - \bar\theta_o) - w_4\bar{\ddot\theta}_o - w_5\bar{\dot\theta}_o \qquad (2\text{-}64)$$

$$\bar{\ddot\theta}_o = -\left[\frac{1}{p}\frac{\alpha_{\ddot\theta o}}{\alpha_{\dddot\theta o}\alpha_\tau}\bar{\dddot\theta}_o\right] = -\frac{1}{p}w_6\bar{\dddot\theta}_o \qquad (2\text{-}65)$$

$$[-\bar{\dot\theta}_o] = \frac{1}{p}\frac{\alpha_{\dot\theta o}}{\alpha_{\ddot\theta o}\alpha_\tau}\bar{\ddot\theta}_o = \frac{1}{p}w_7\bar{\ddot\theta}_o \qquad (2\text{-}66)$$

$$\bar{\theta}_o = -\left[\frac{1}{p}\frac{\alpha_{\theta o}}{\alpha_{\dot\theta o}\alpha_\tau}\bar{\dot\theta}_o\right] = -\frac{1}{p}w_8\bar{\dot\theta}_o \qquad (2\text{-}67)$$

$$u = -\left[7.74\frac{\alpha_{\theta i}}{\alpha_u}\bar{x} - 5.87\frac{\alpha_{\dot\theta o}}{\alpha_u}\bar{\dot\theta}_o - \frac{\alpha_{\ddot\theta o}}{\alpha_u}\bar{\ddot\theta}_o\right] \qquad (2\text{-}68)$$

$$= -w_{11}\bar{x} + w_{12}\bar{\dot\theta}_o + w_{10}\bar{\ddot\theta}_o$$

The calculated values are in Table 2-4 the diagram of actual setting is in Fig. 2-6b, and the various plots are in Fig. 2-20 through Fig. 2-27.

TABLE 2-4 Data for Analog Computer Setup of Fig. 2-6b

$W_1=$	1		$R_{fA} = 1$	$R_1=0.5$	$a_1=0.5$
$W_2=$	1		$R_{fA} = 1$	$R_2=0.5$	$a_2=0.5$
$W_3=$	0.5		$R_{fB} = 1$	$R_3=1$	$a_3=0.5$
$W_4=$	0.337	(0.663)	$R_{fB} = 1$	$R_4=1(0.5)$	$a_4=0.33(0.332)$
$W_5=$	0.18	(2.65)	$R_{fB} = 1$	$R_5=1(0.1)$	$a_5=0.18(0.265)$
$W_6=$	6		$C_{fe}=1.04$	$R_6=0.1$	$a_6=0.624$
$W_7=$	0.5		$C_{fd}=1.028$	$R_7=1$	$a_7=0.515$
$W_8=$	0.666		$C_{fE}=1.08$	$R_8=1$	$a_8=0.72$
$W_9=$	1		$R_{fF}= 1$	$R_9=0.5$	$a_9=0.5$
$W_{10}=$	0.05		$R_{fG}=1$	$R_{10}= 1$	$a_{10}=0.05$
$W_{11}=$	1.16		$R_{fG}=1$	$R_{11}= 0.5$	$a_{11}=0.58$
$W_{12}=$	0.587		$R_{fG}=1$	$R_{12}= 1$	$a_{12}=0.587$

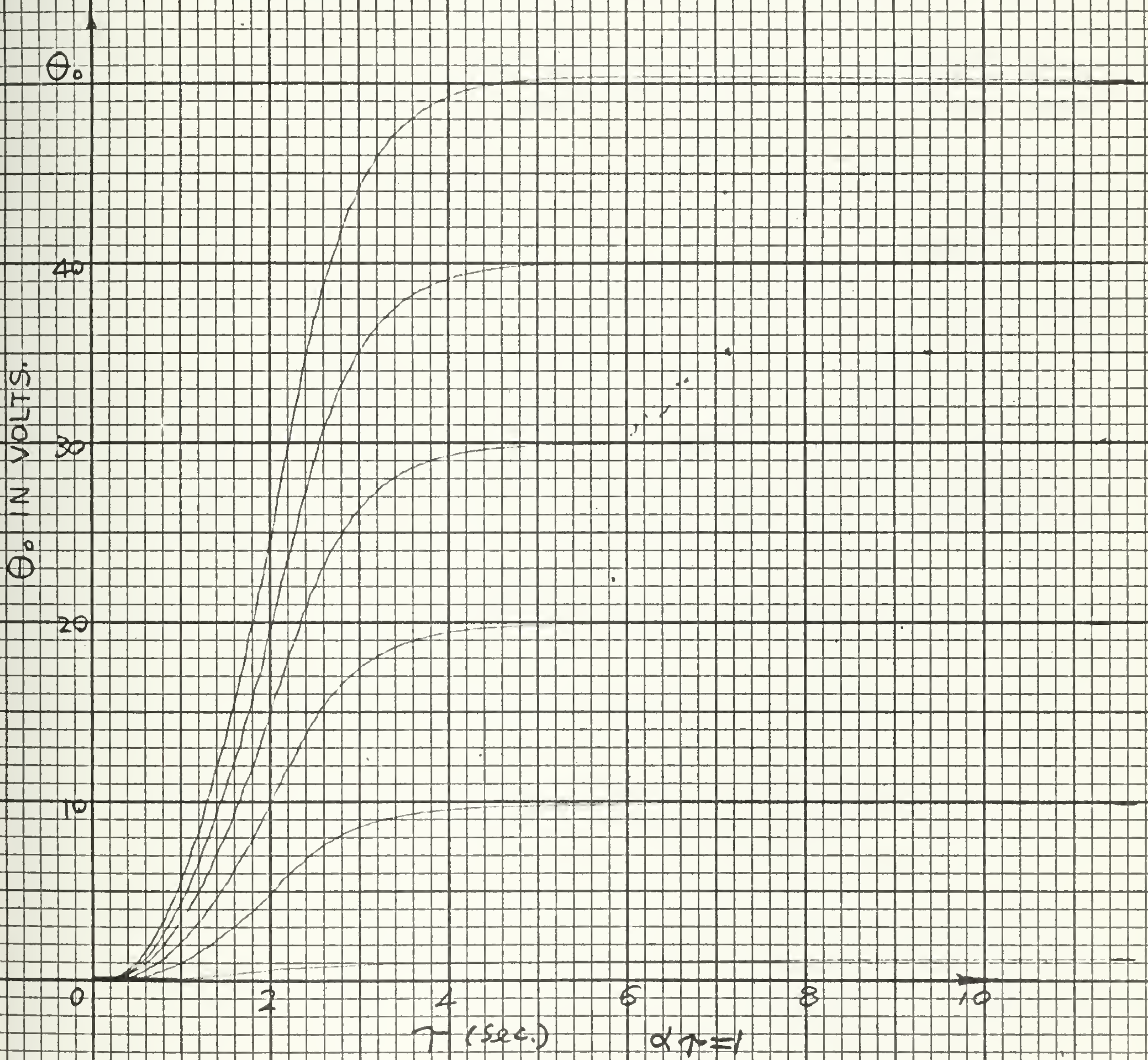


Fig. 2-20 Transient Response for Different Magnitudes of Step Inputs

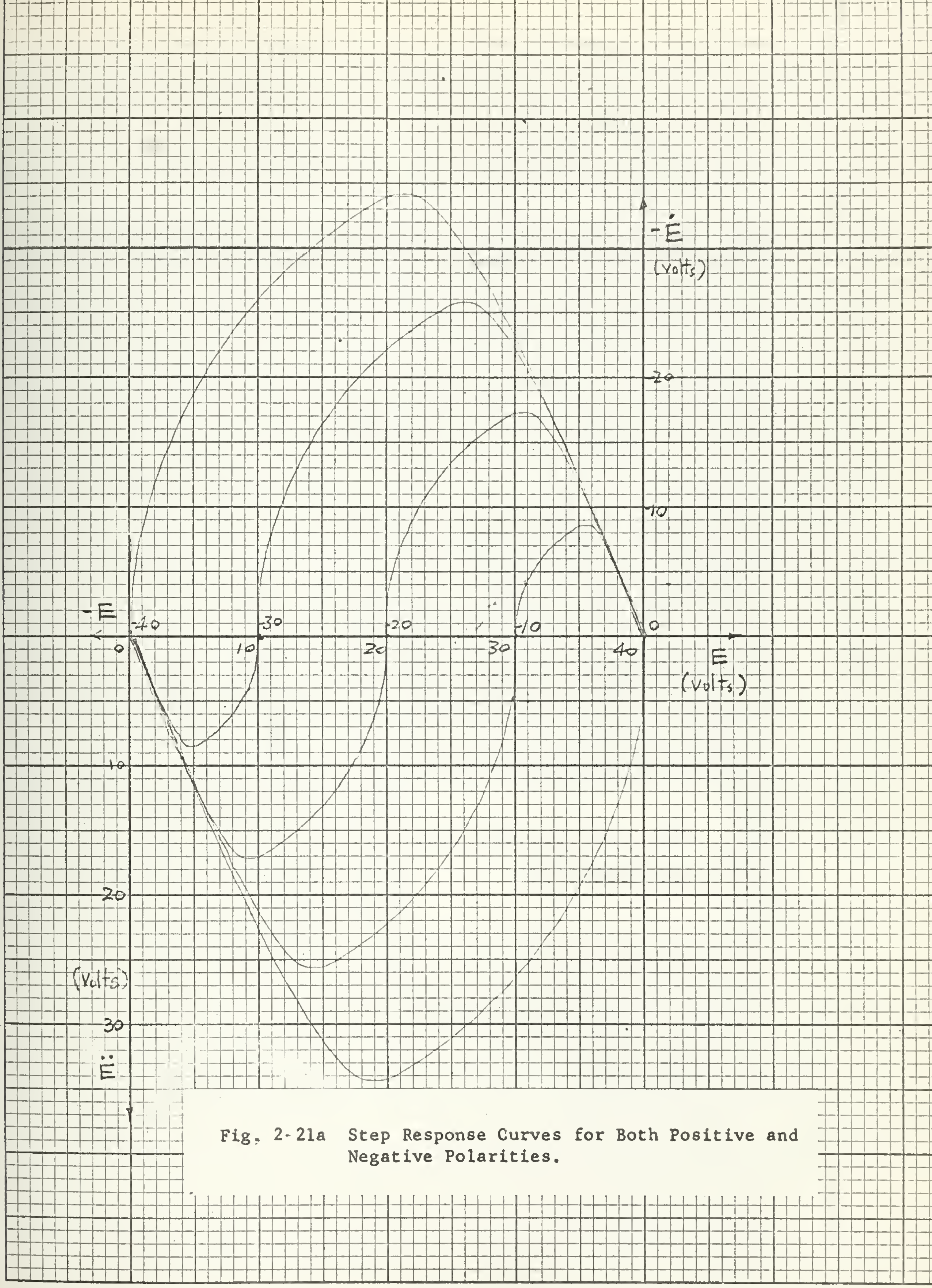


Fig. 2-21a Step Response Curves for Both Positive and Negative Polarities.

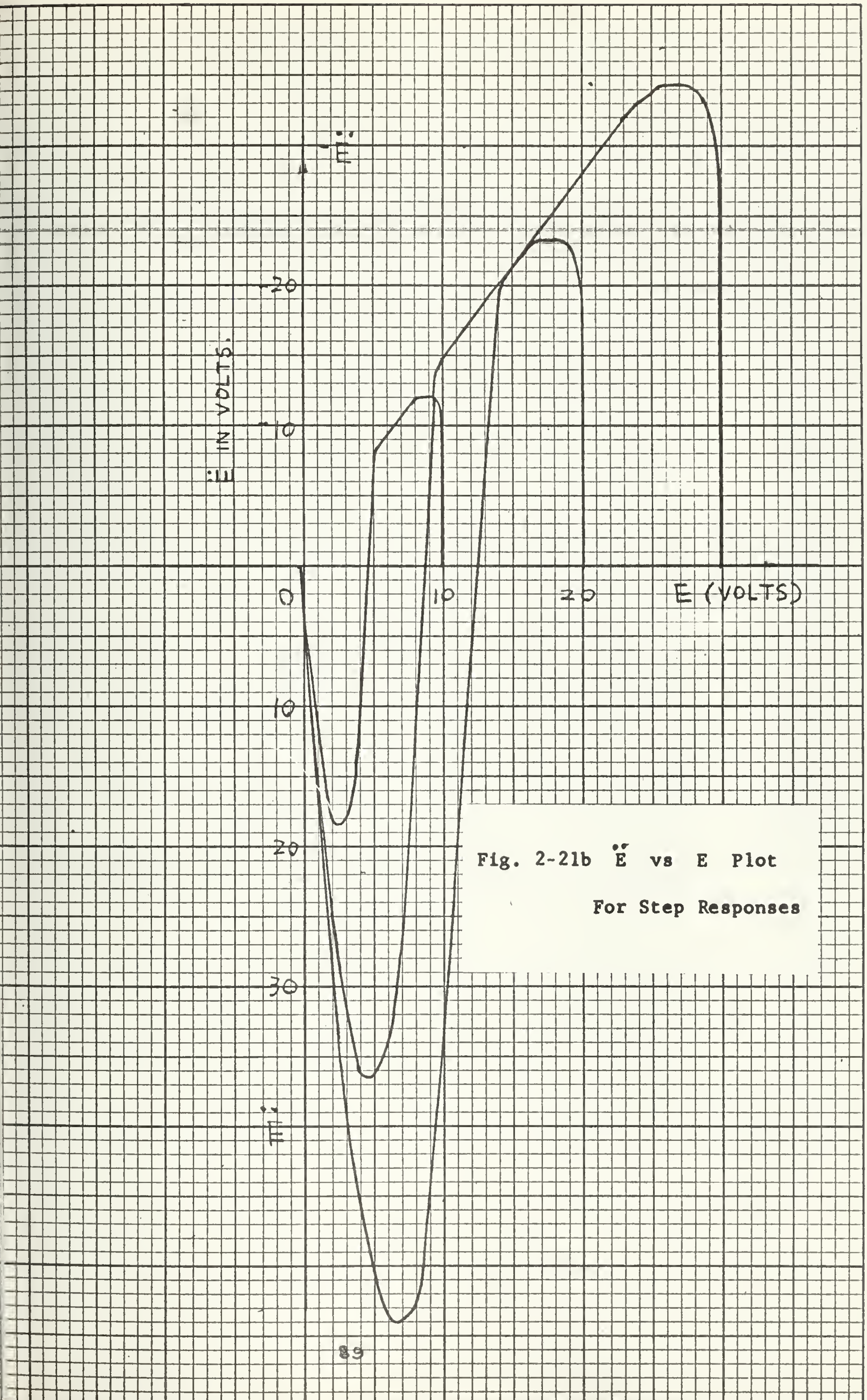


Fig. 2-21b $\ddot{E}$ vs E Plot

For Step Responses

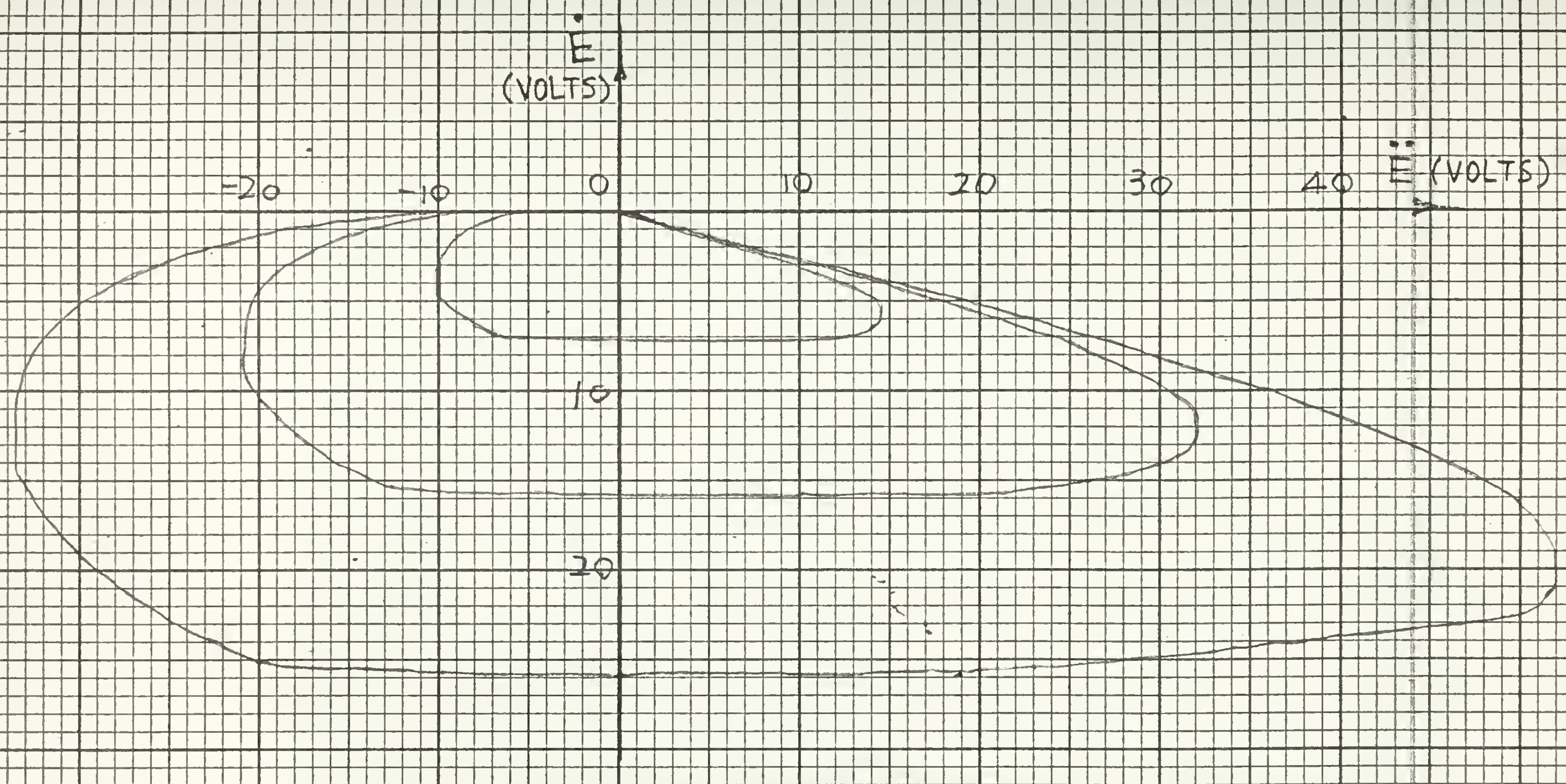


Fig. 2-21c $\ddot{E}$ vs. $\dot{E}$ Plot for Step Responses.

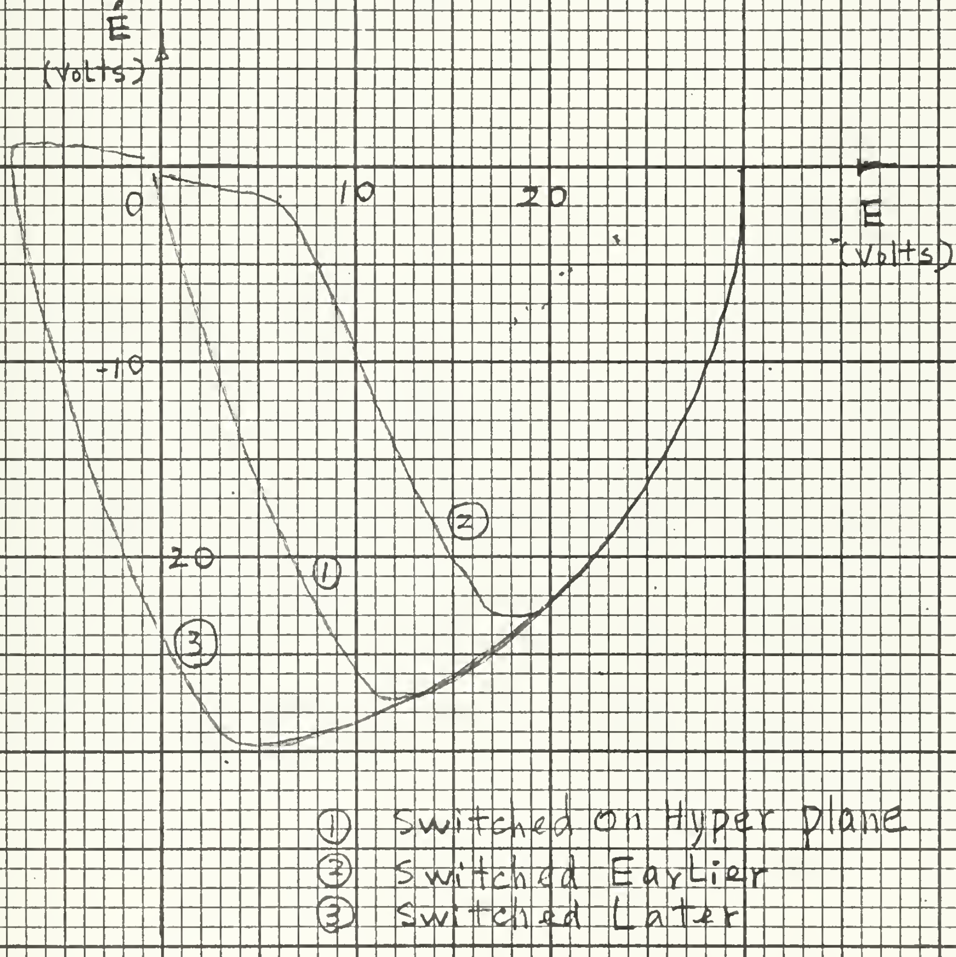


Fig. 2-22a Step Responses for Different Switching Time

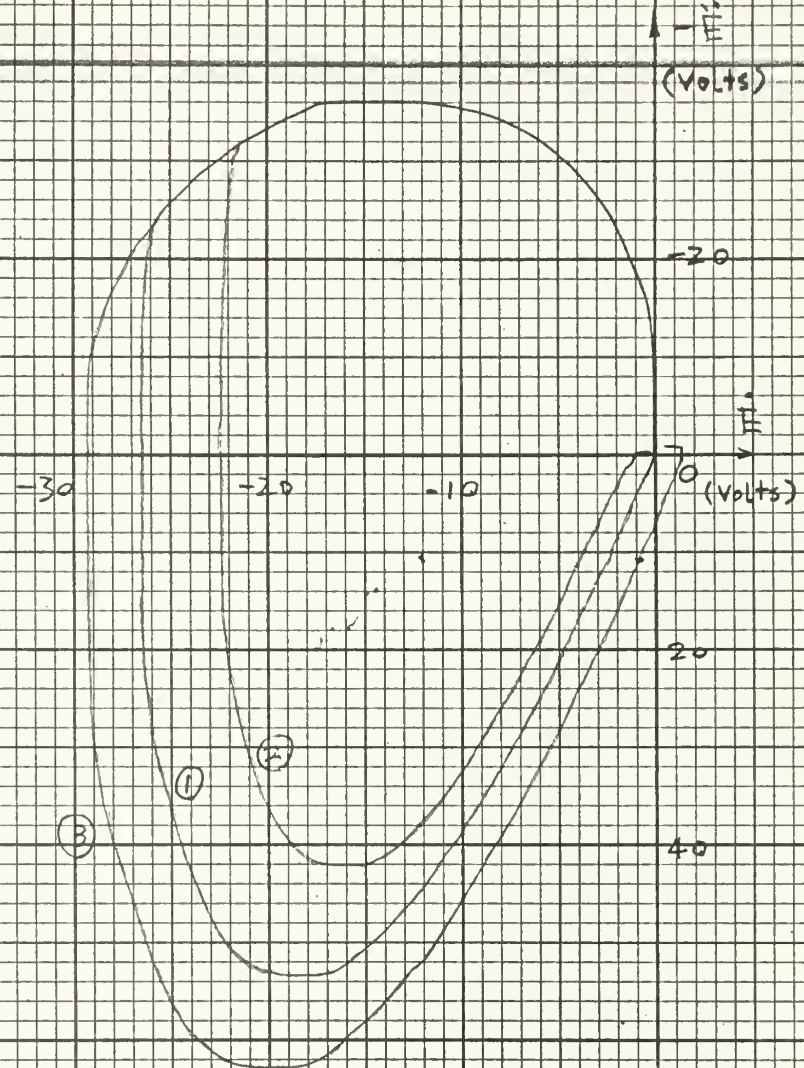


Fig. 2-22b $\ddot{E}$ vs. $\dot{E}$ Plots for Different Switching Time
(Step Input 30^V)

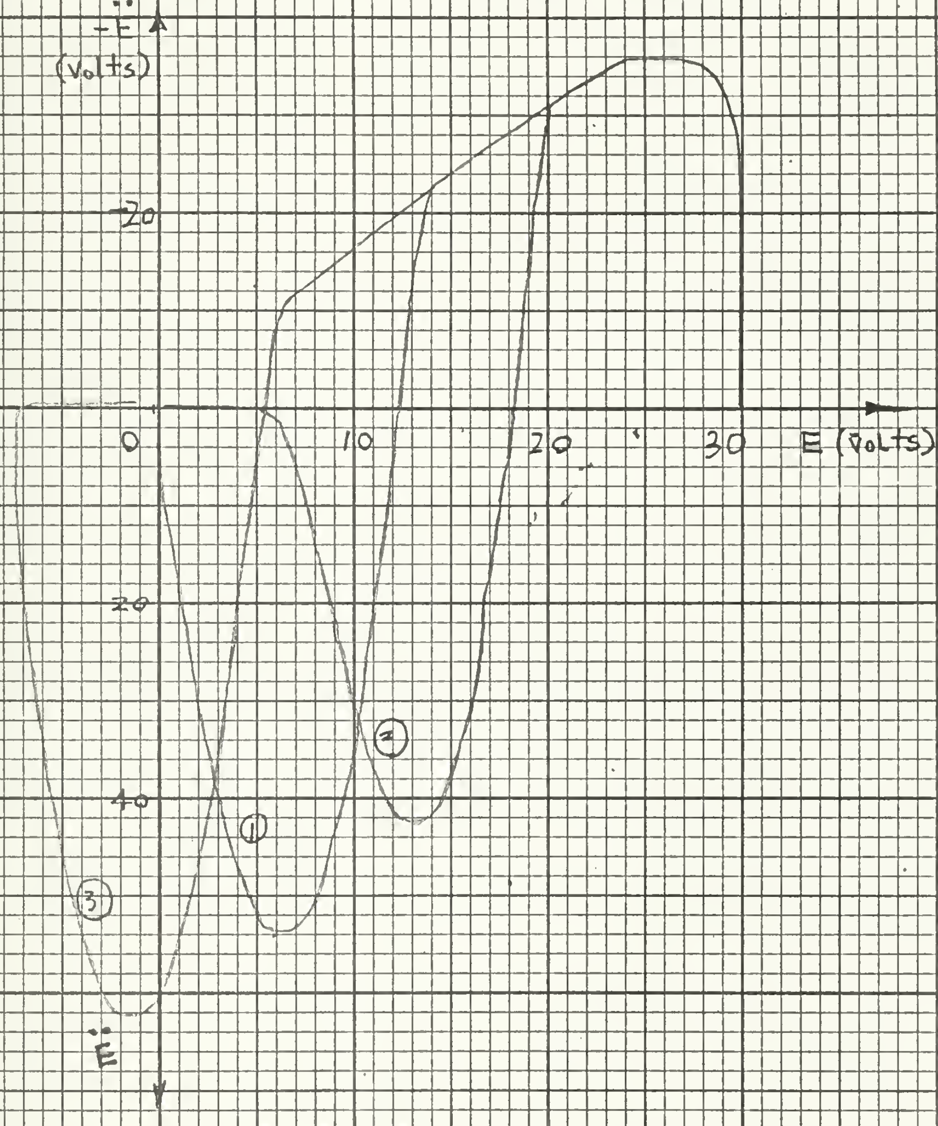


Fig. 2-22c $\ddot{E}$ vs. E Plot for Different Switching Time

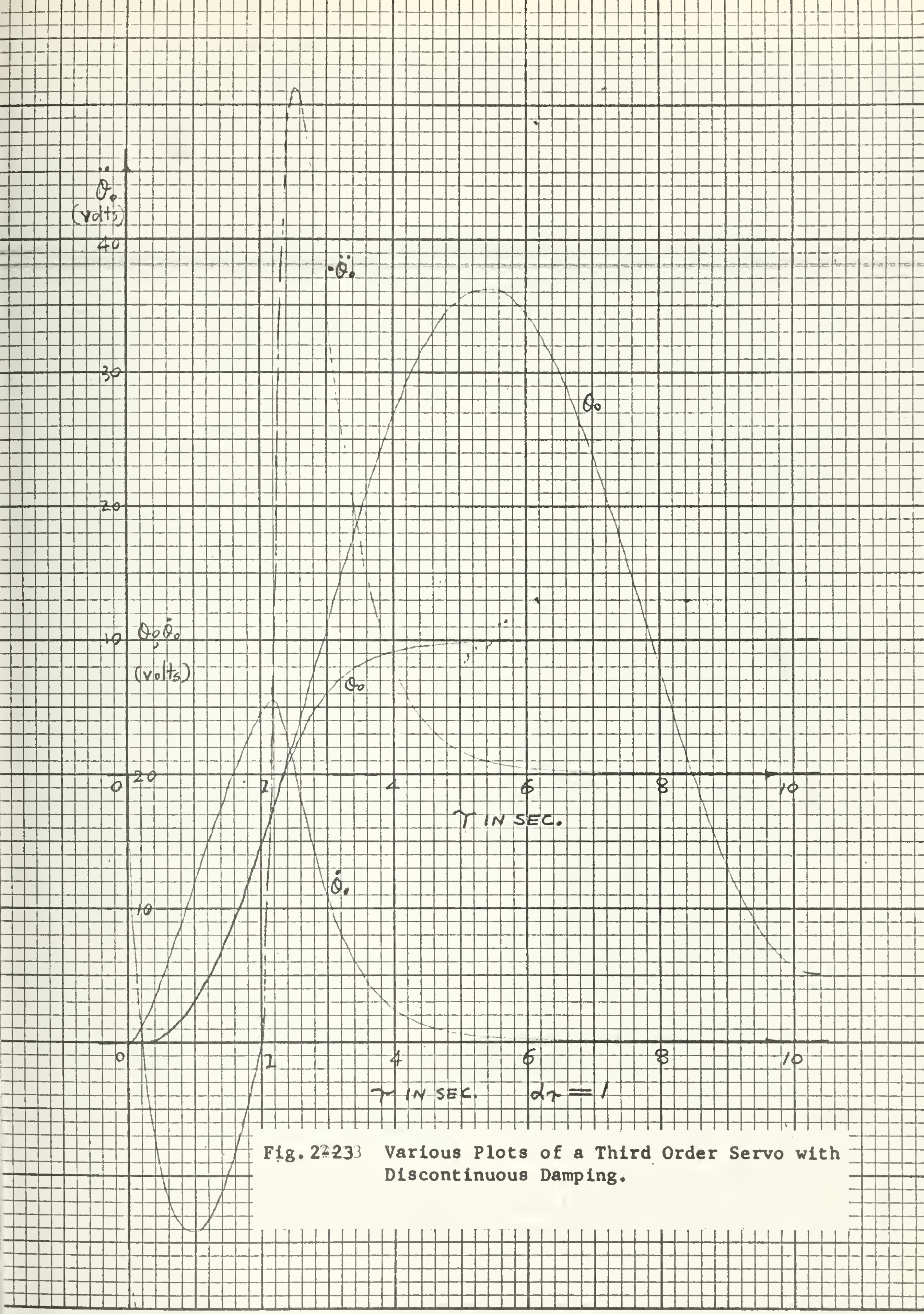


Fig. 2.23 Various Plots of a Third Order Servo with Discontinuous Damping.

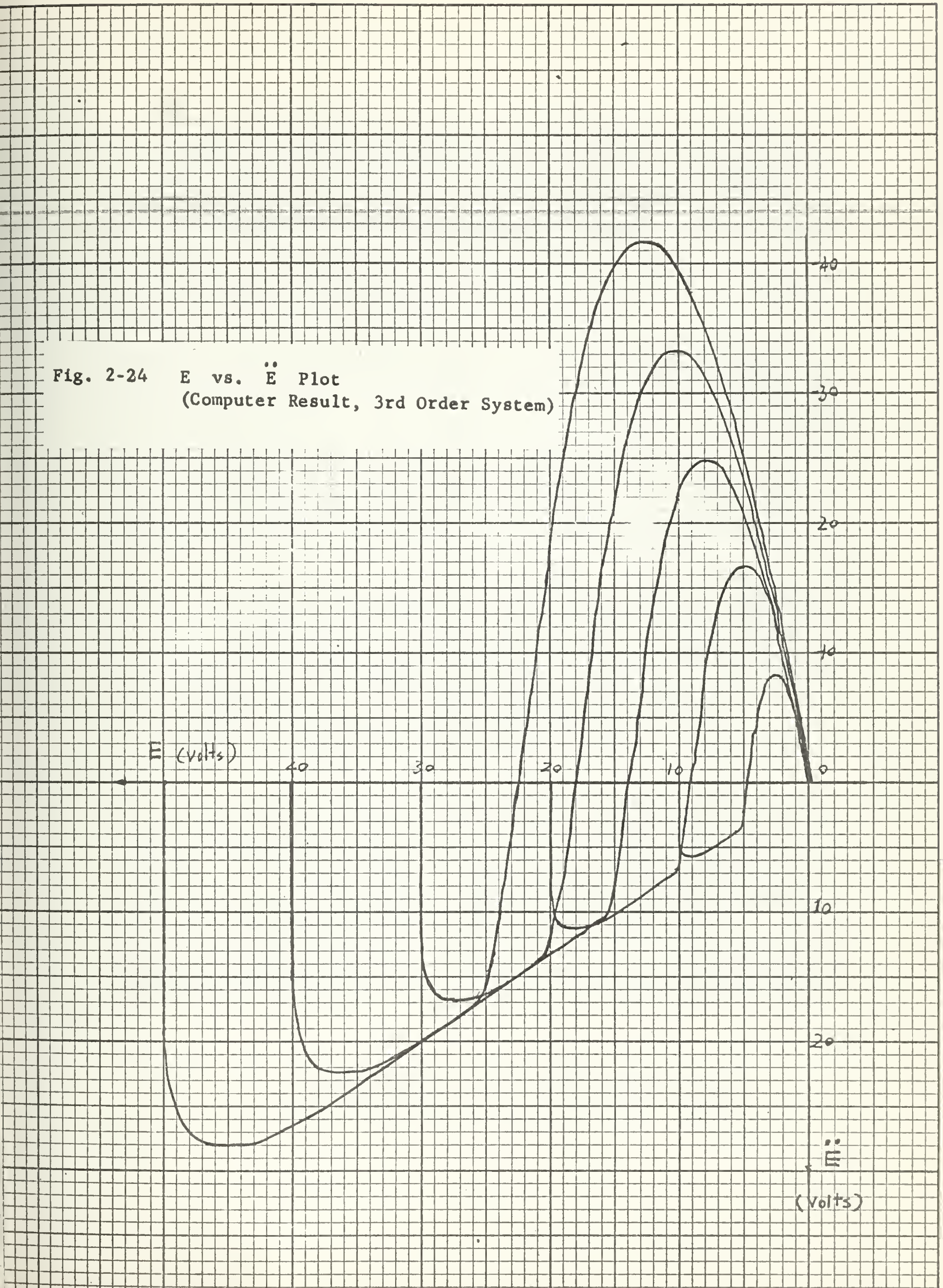


Fig. 2-24 E vs. Ë Plot
(Computer Result, 3rd Order System)

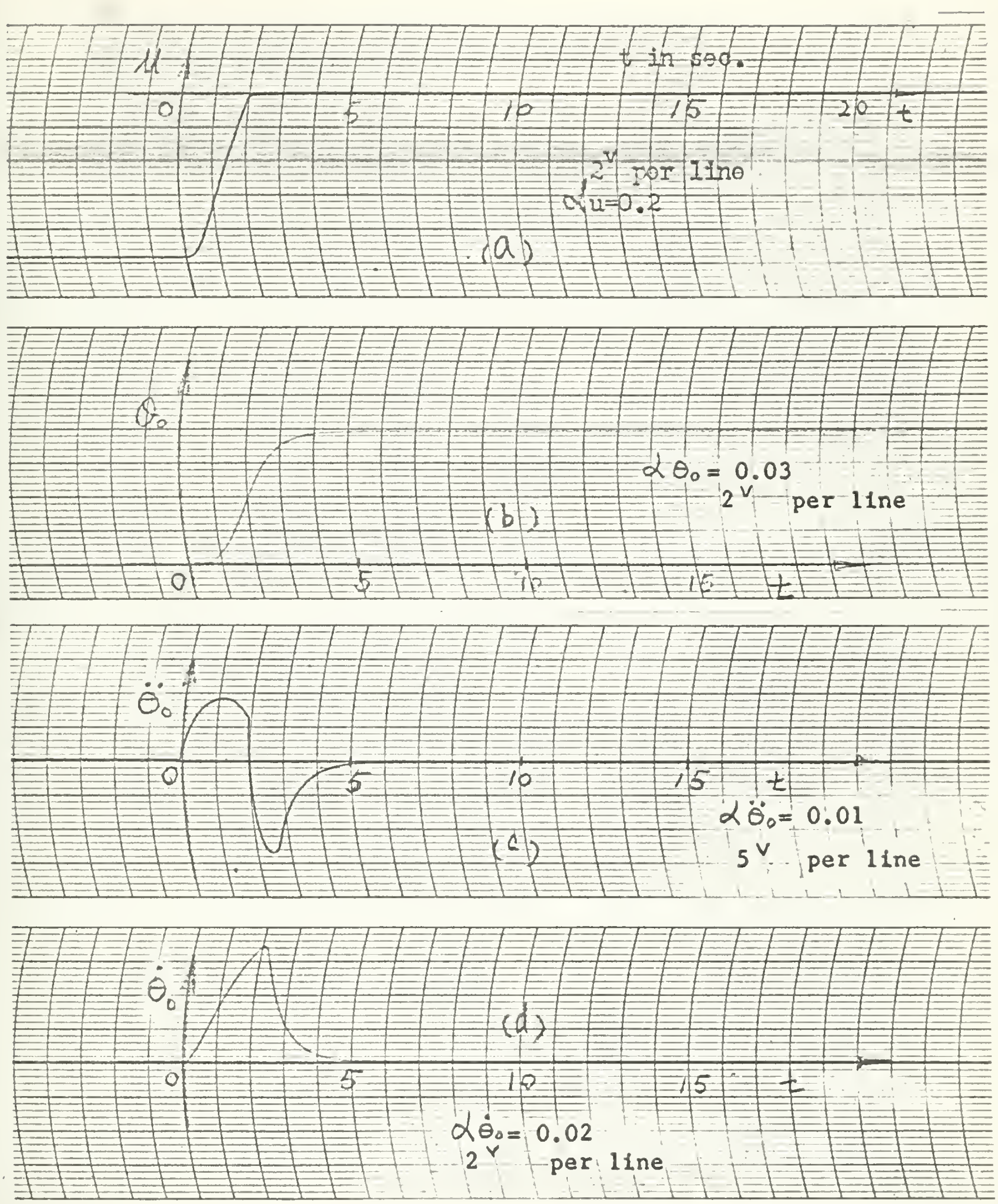


Fig. 2-25 Illustrations; (a) u Signal stay at zero after switching. (b), (c), and (d) various signals for a step input (Computer Results).

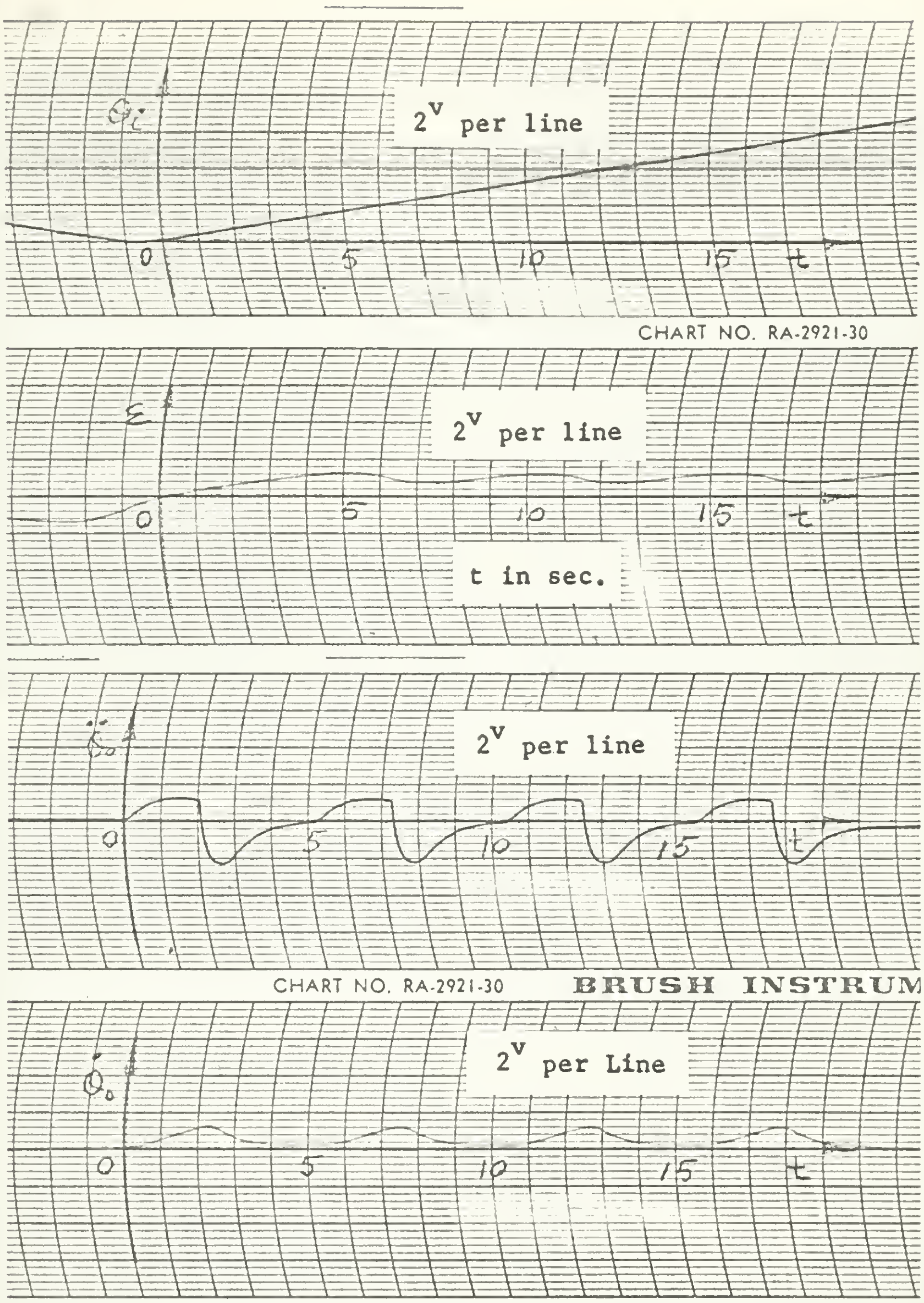


Fig. 2-26 Various Wave Forms for a ramp input
(Computer Results).

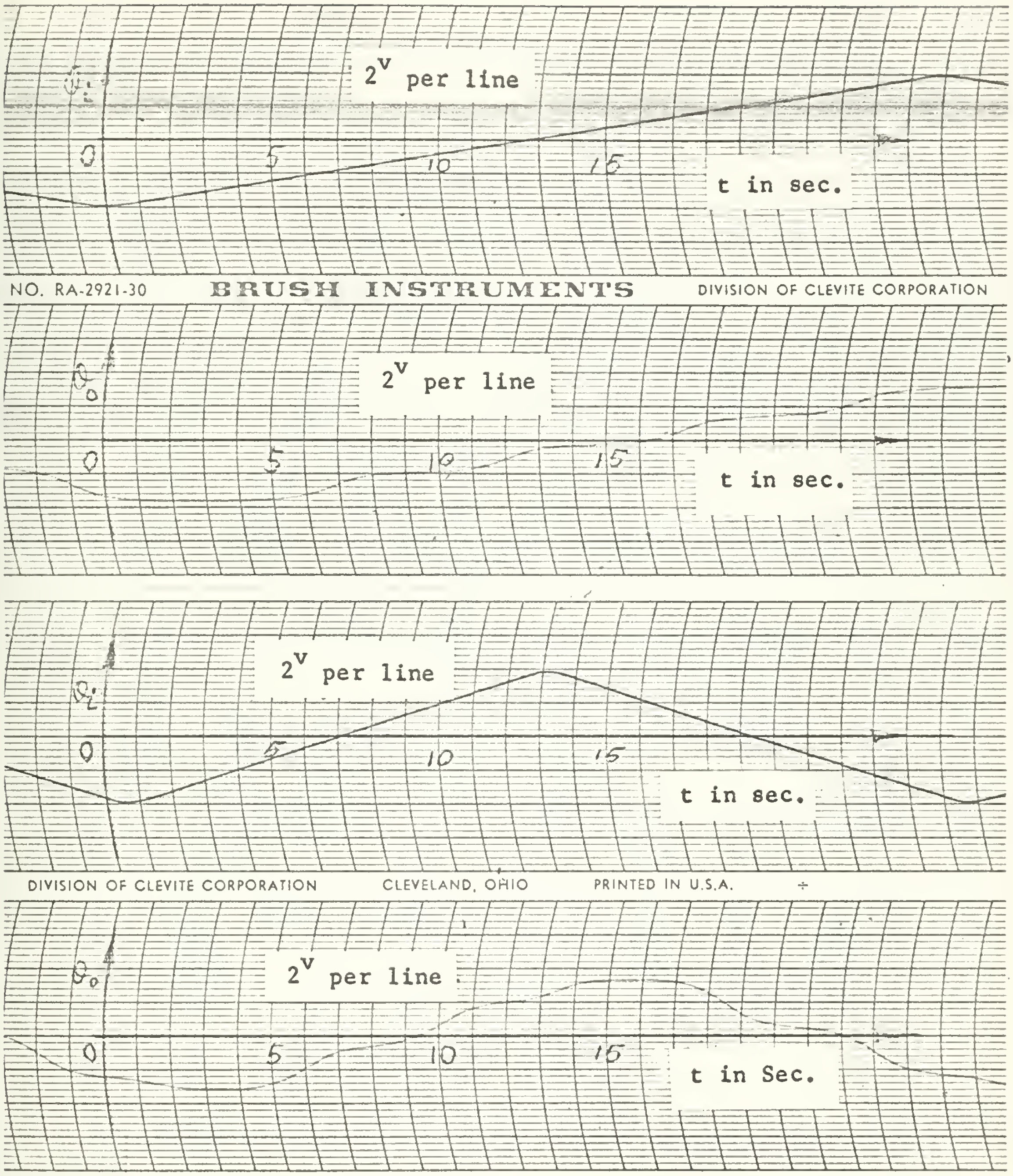


Fig. 2-27 Transient Response for Triangular Inputs (Computer Results).

SECTION IV - FOURTH ORDER SYSTEMS

(A) Calibration and Parameter Evaluation.

The block diagram is in Fig. 2-28. The various results of calibrations are in Fig. 2-29 to Fig. 2-32. In Fig. 2-33 and Fig. 2-34 the frequency response curve shows that the amplidyne has quadratic poles at $\omega = 16.5, \zeta = 0.7$ and Fig. 2-35 shows the other pole is at $\omega = 8$. In Fig. 2-36 complete root loci have been plotted by arbitrarily selecting two complex poles near the imaginary axis $p_1 = -0.35 + j8$, $p_2 = -0.35 - j8$ the locations of the other two poles are decided by the value of the gain 23774 at p_1 and p_2, and they are located at $-15.15 \pm j11.89$.

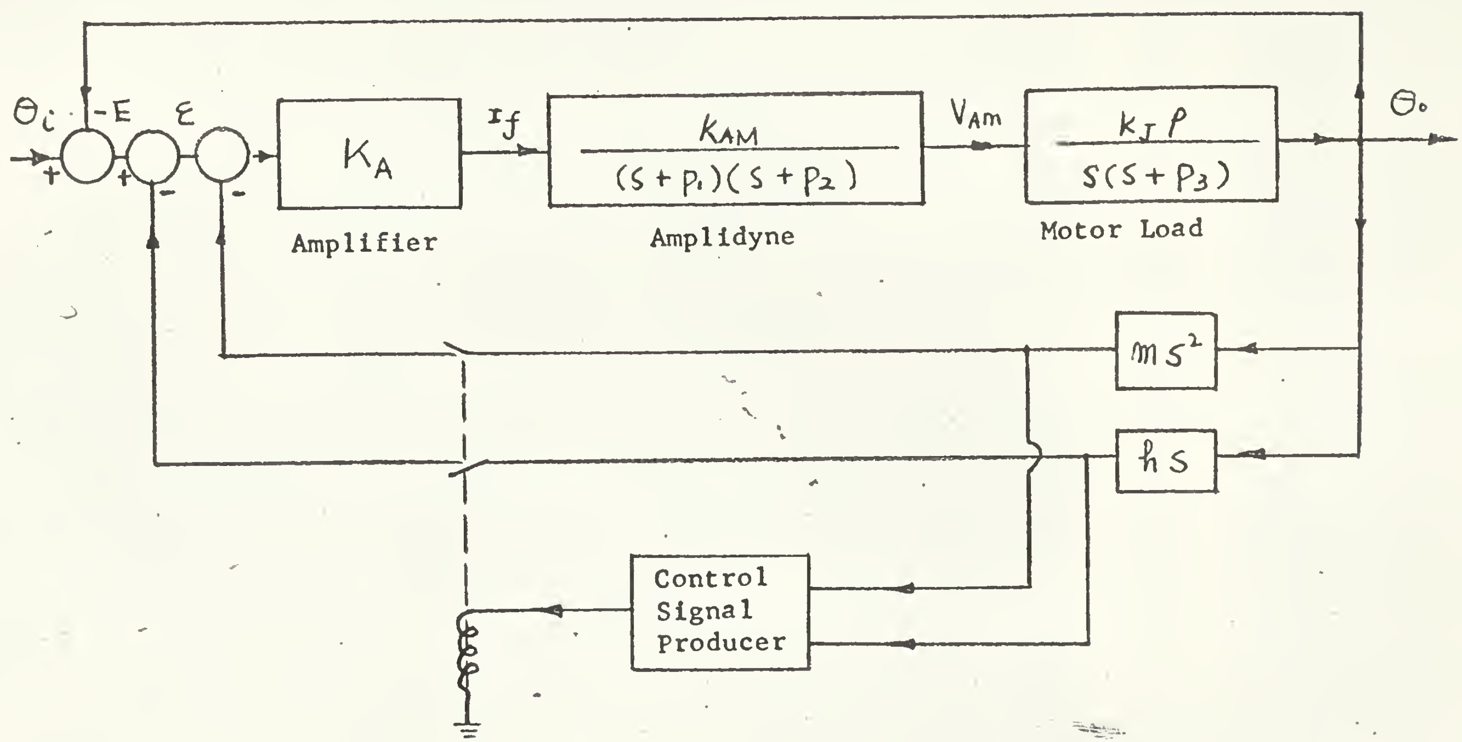
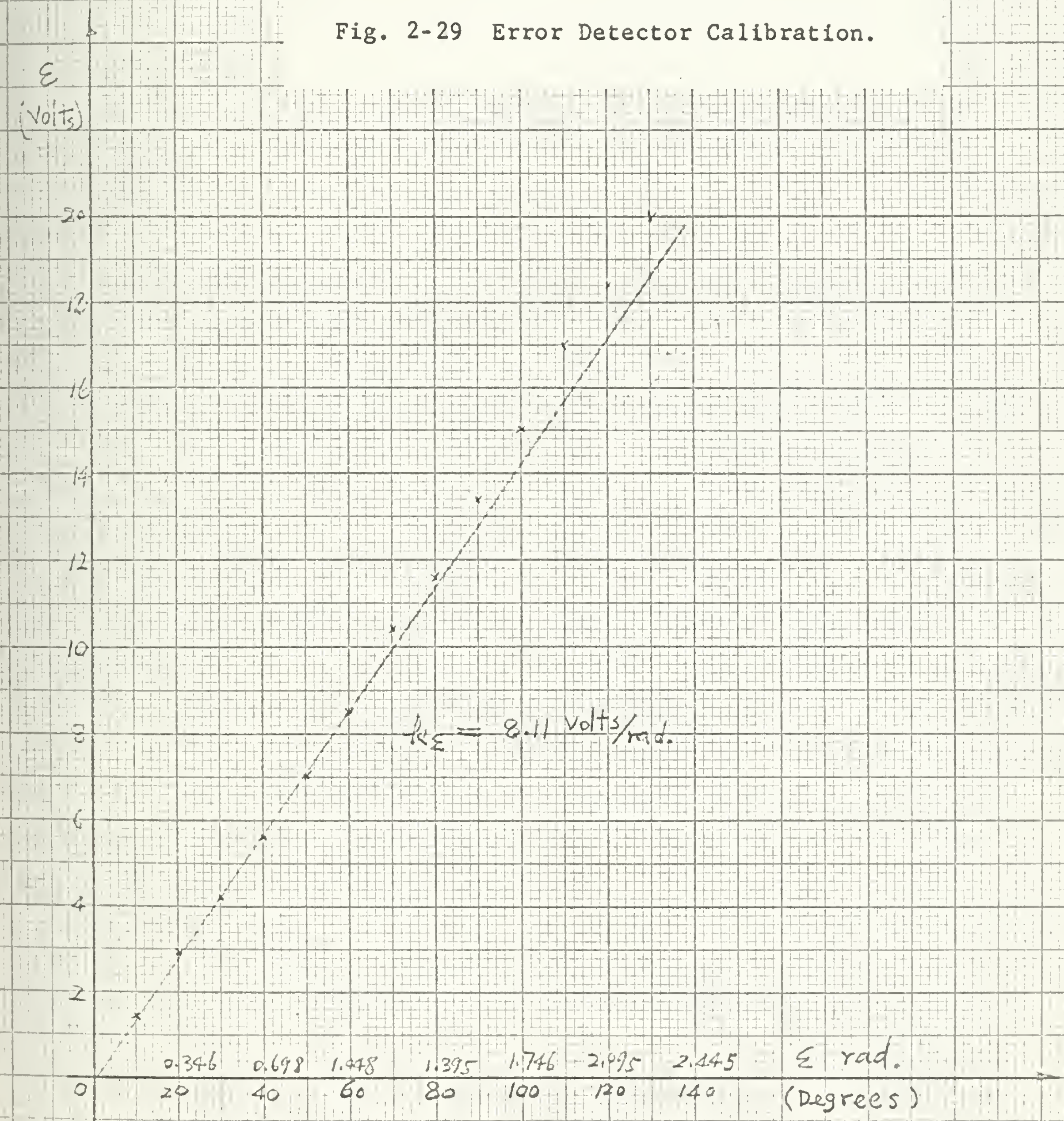


Fig. 2-28 Block Diagram of a Real Fourth Order Servo System with Discontinuous Damping Circuits

Fig. 2-29 Error Detector Calibration.



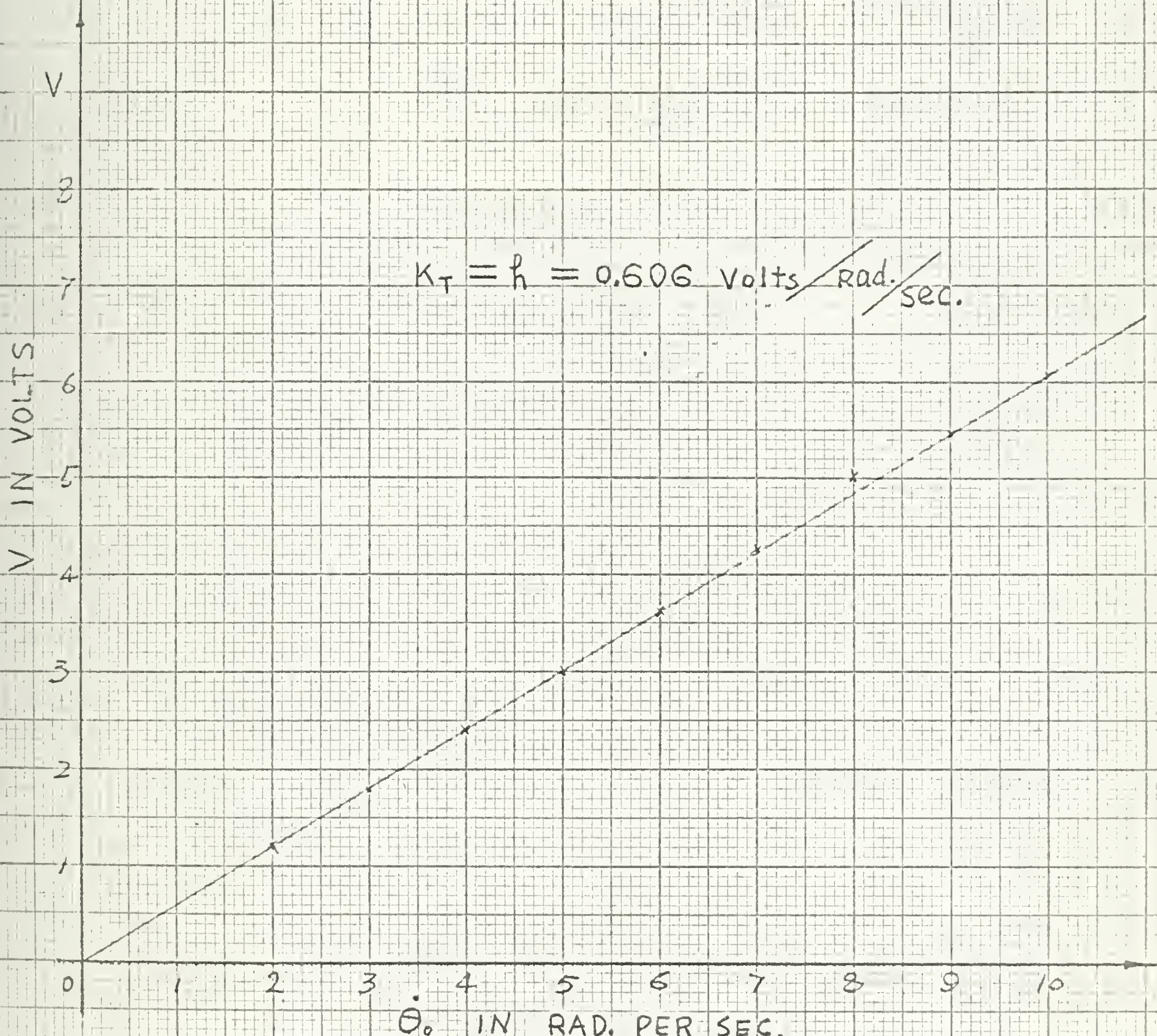


Fig. 2-30 Tach. Calibration

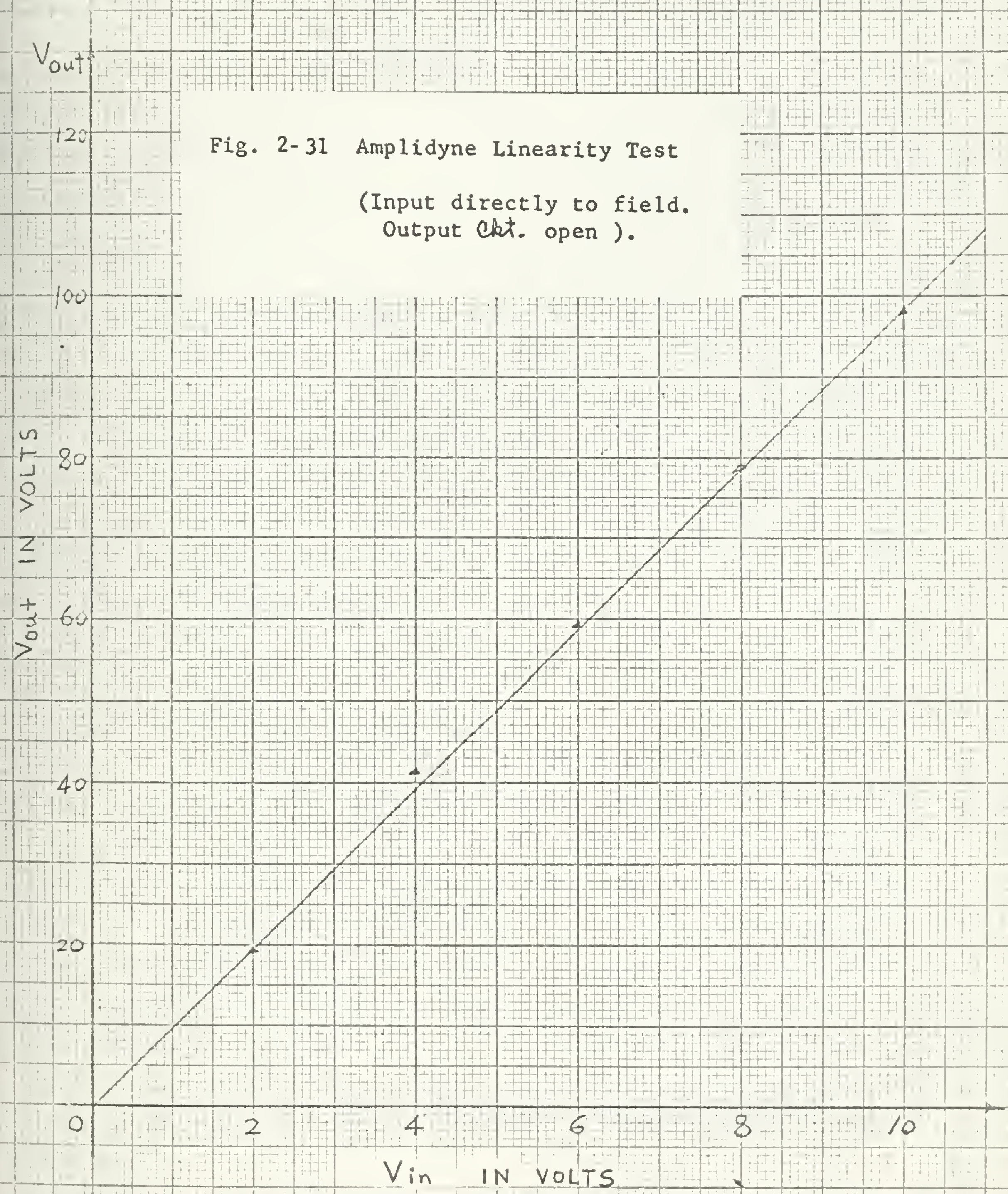


Fig. 2-31 Amplidyne Linearity Test

(Input directly to field. Output Ckt. open).

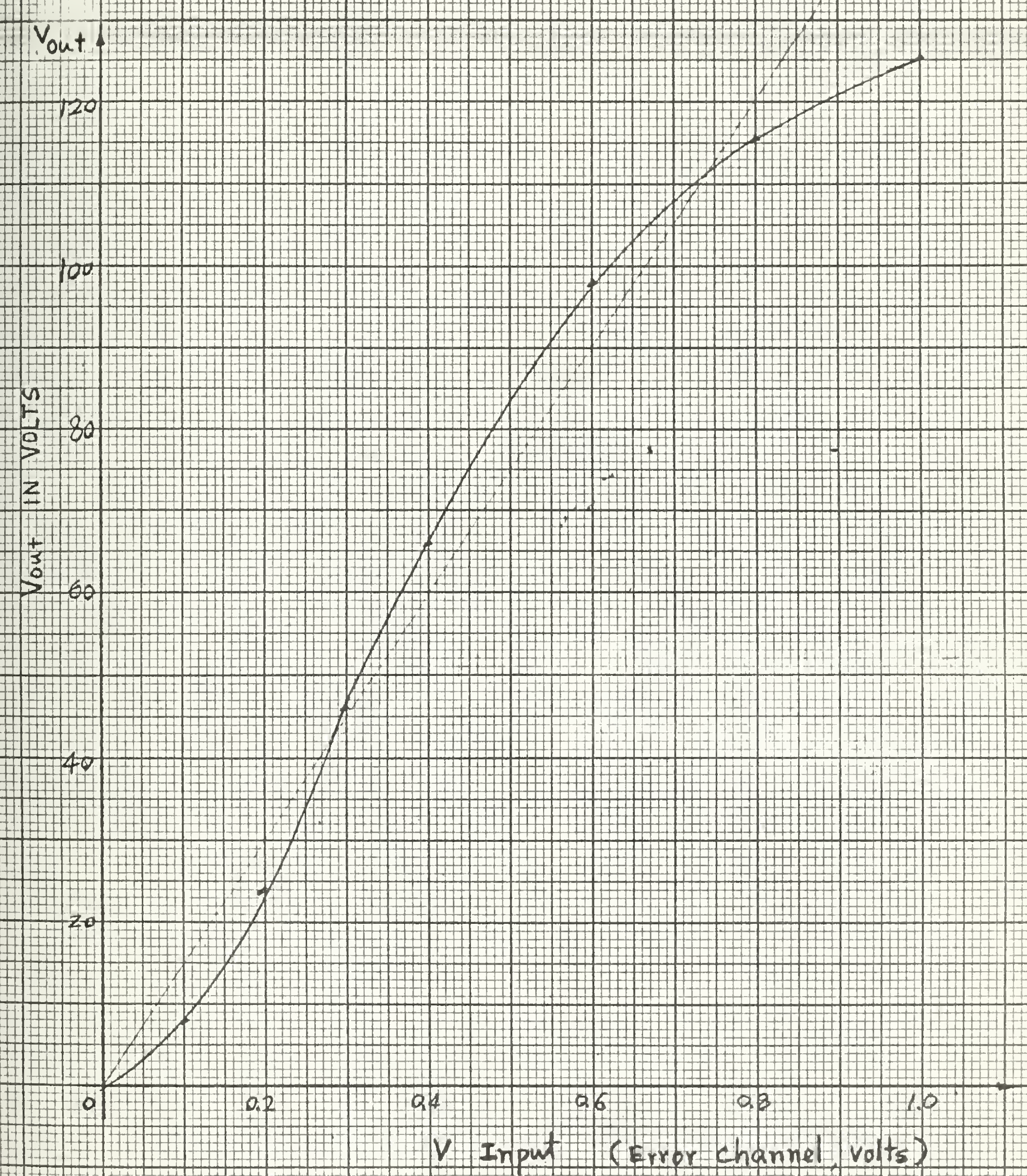


Fig. 2-32 Amplidyne Linearity Test with Motor Running.

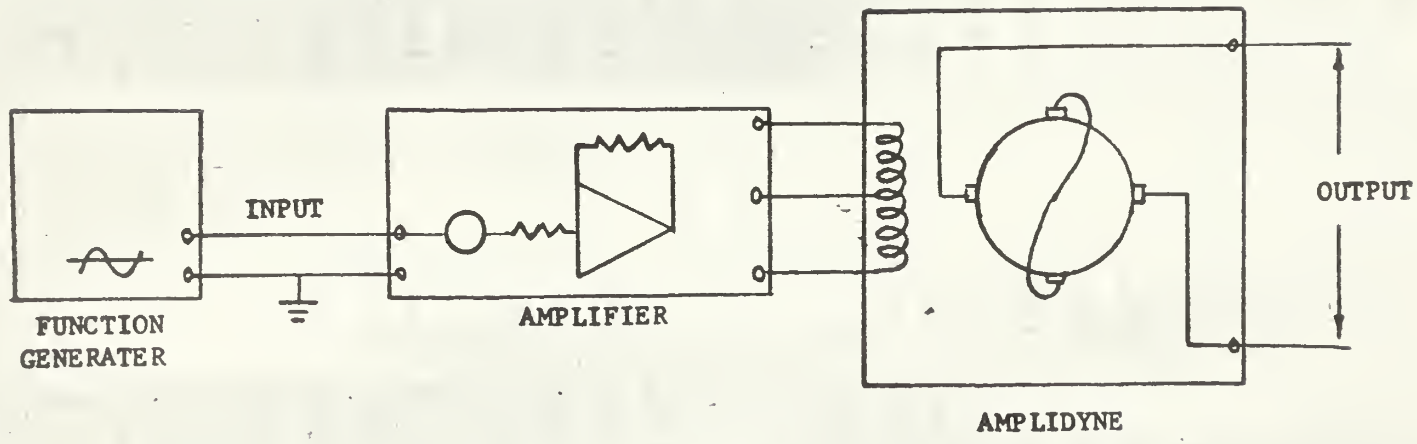


Fig. 2-33 Schematic Block Diagram for the Open Loop Frequency Response Test of Amplidyne.

TABLE 2-5 Frequency Response for Amplidyne Only
(with 0.5 Y Input to X channel)

f cyc/sec.	θ_o/θ_i		db	ω rad/sec.
1.0	0.94		-0.56	6.28
1.25	0.936		-0.594	7.85
1.5	0.928		-0.67	9.42
1.8	0.898		-0.956	11.33
2.0	0.864		-1.28	12.56
2.4	0.788		-2.08	15.07
2.7	0.697		-3.15	16.95
3.0	0.618		-4.2	18.84
3.[illegible]	0.491		-6.2	22.0
4.0	0.41		-7.76	25.1
4.5	0.331		-9.614	28.25
5.0	0.2715		-11.34	31.4
5.5	0.2255		-12.95	34.55
6.0	0.1927		-14.32	37.6
6.5	0.1648		15.67	40.8
7.0	0.1418		-16.974	44
7.5	0.1212		18.334	47
8.0	0.1042		-19.65	50.1
8.5	0.0934		-20.62	53.2
9.0	0.0825		-21.68	56.5
9.5	0.0764		-22.36	59.6
10.0	0.0694		-23.2	62.8

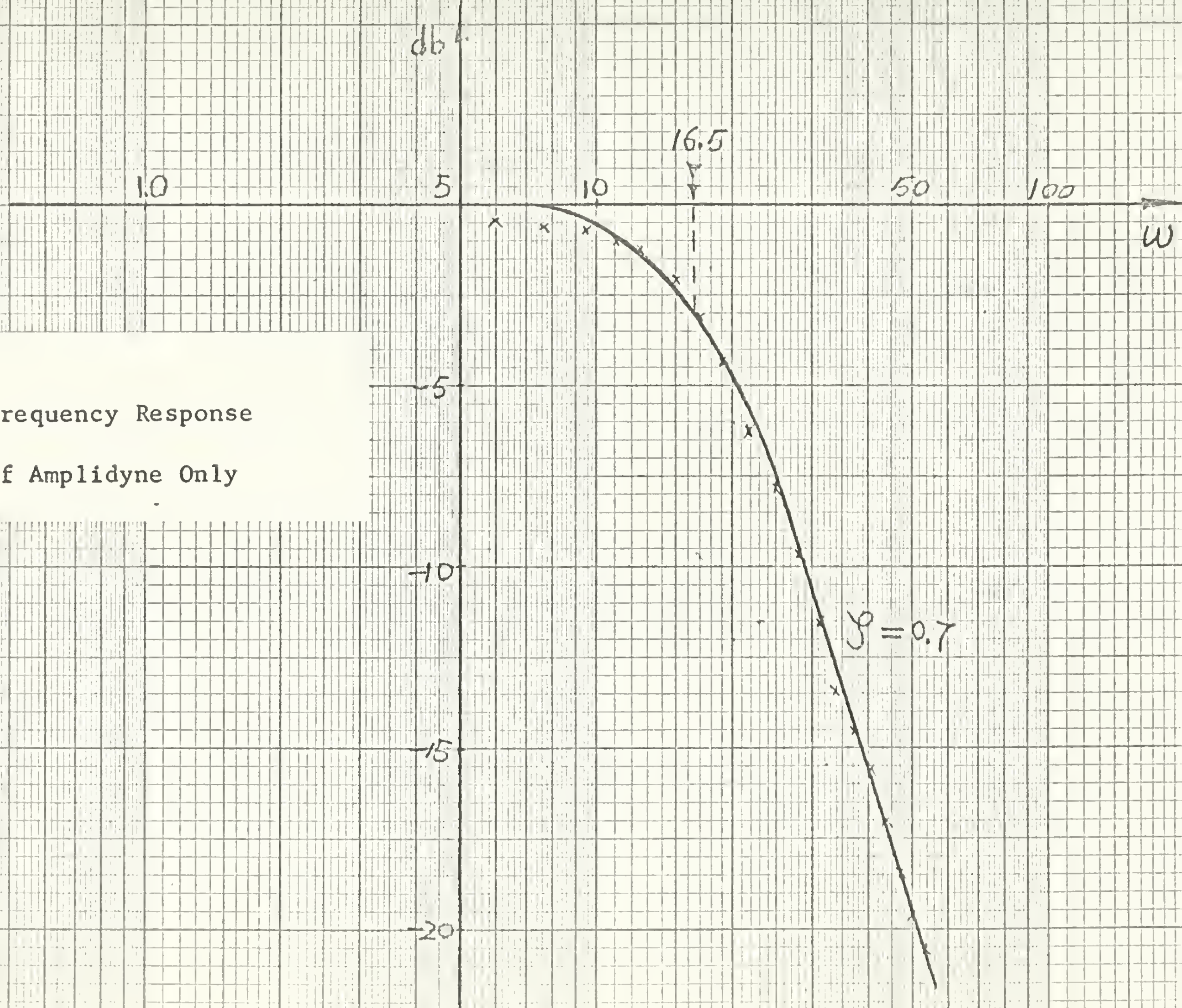


Fig. 2-34 Frequency Response of Amplidyne Only

TABLE 2-6 Frequency Response of Open Loop System (with 0.5V Input to X channel, Output from Tachometer)

f cyc/sec.	θ_o/θ_i	ω	db
0.01	1	0.0628	0
0.4	0.948	2.513	-0.48
0.6	0.94	3.77	-0.75
1.0	0.901	6.28	-0.92
1.25	0.858	7.85	-1.34
1.5	0.82	9.42	-1.74
1.8	0.715	11.3	-2.92
2.0	0.655	12.56	-3.68
2.4	0.526	15.07	-5.59
2.7	0.4255	16.95	-7.44
3.0	0.354	18.84	-9.03
3.5	0.243	22.0	-12.3
4.0	0.176	25.1	-15.094
4.5	0.132	28.25	-17.6
5.0	0.1034	31.4	-19.71
5.5	0.079	34.55	-22.06

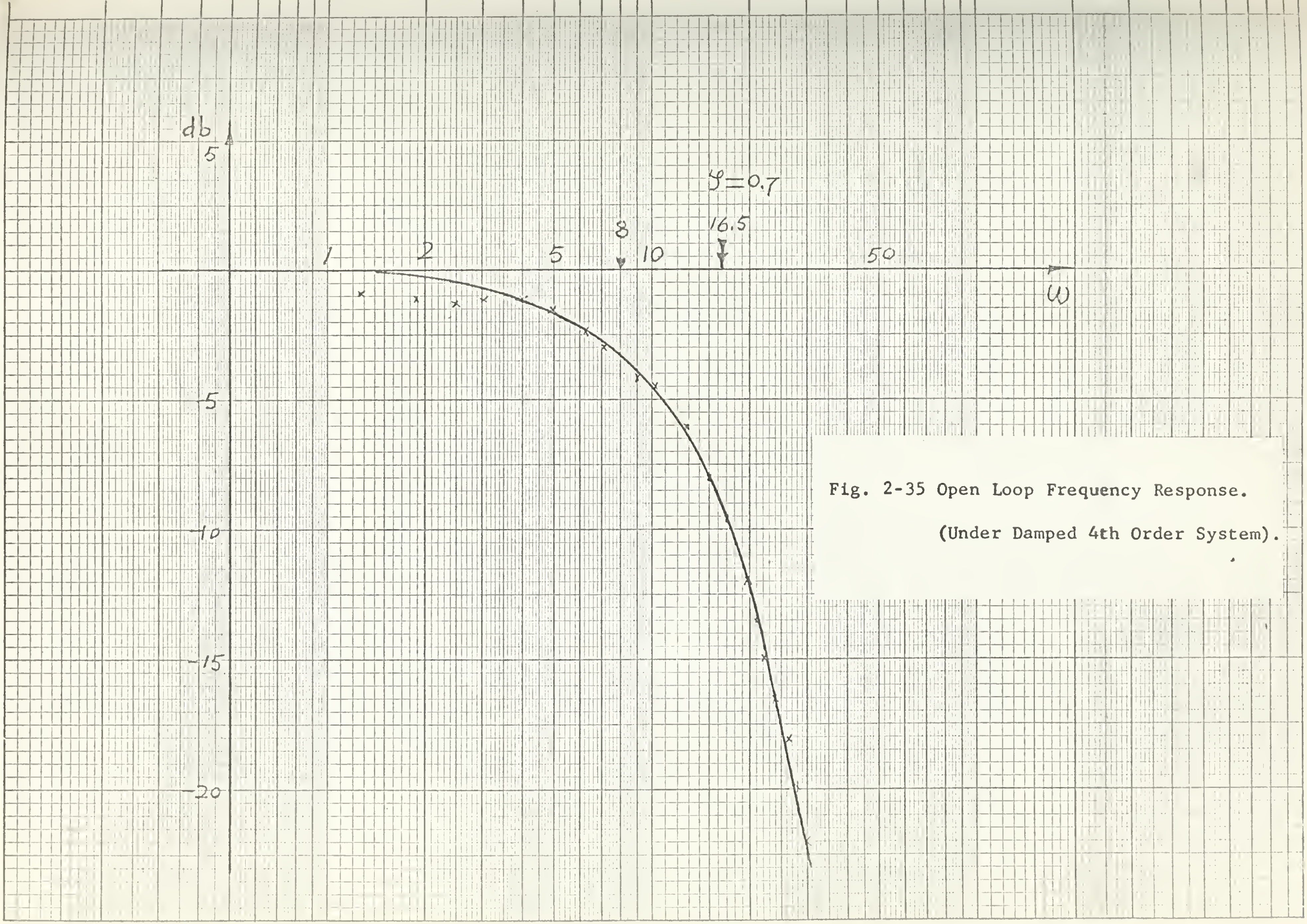


Fig. 2-35 Open Loop Frequency Response.

(Under Damped 4th Order System).

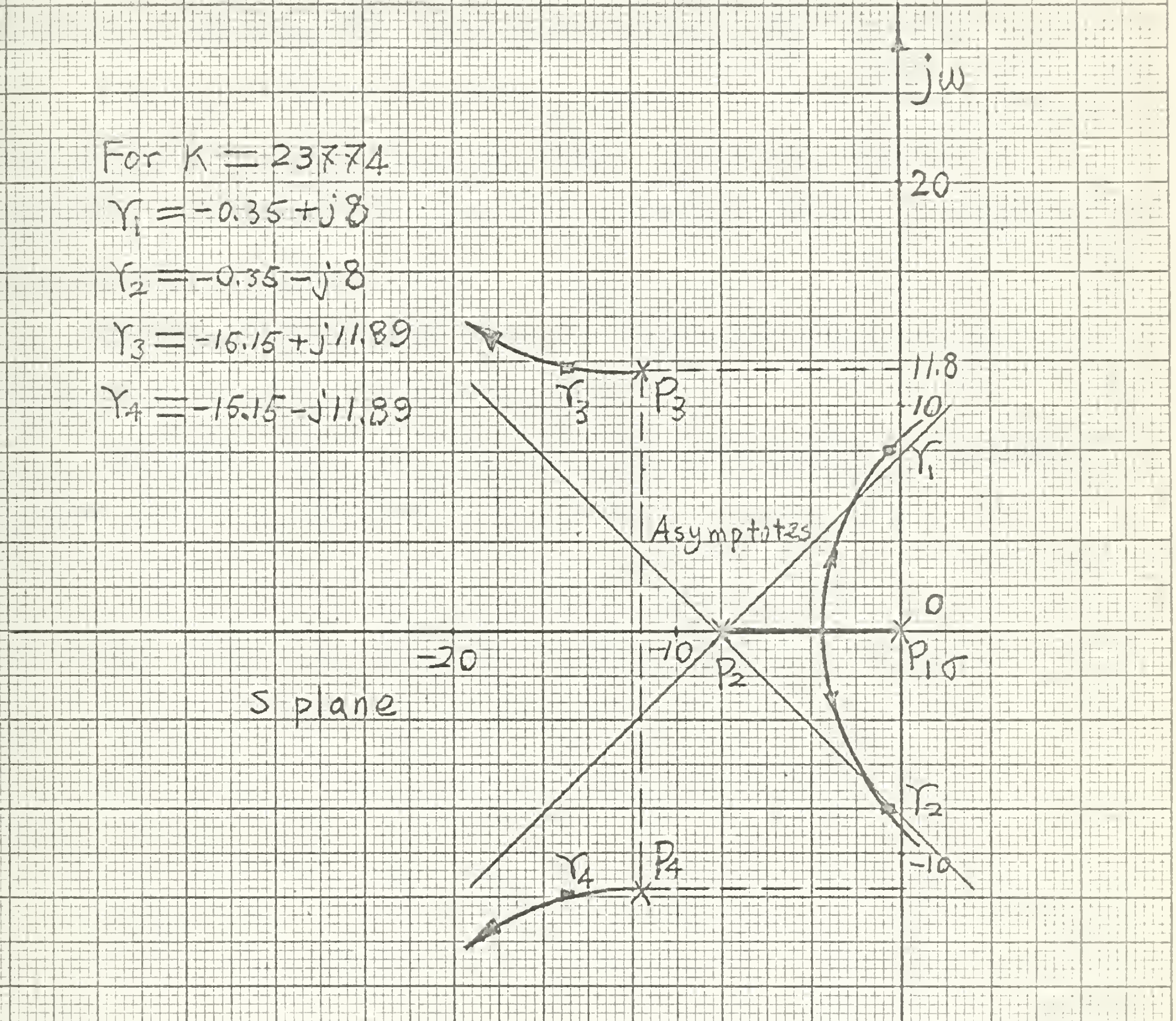


Fig. 2-36 Root Loci for

$$G(s) = \frac{K}{S(S+8)(S+11.55+j11.8)(S+11.55-j11.8)}$$

(B) Root Locus Analysis for the 4th Order Physical System by Feeding Back $\dot{\theta}_o$ and $\ddot{\theta}_o$ only.

From Fig. 2-37a, if the zero is too close to the origin, the system will have a low frequency oscillation and the response is very slow. This is due to the large $\ddot{\theta}_o$ feedback or too small $\dot{\theta}_o$ feedback.

From Fig. 2-37b, if h is too large and m is too small, then one of the zeros will be far from the origin. By the root locations as indicated in this figure, the system will have a high frequency oscillation due to the small attenuation factor.

Therefore, the best case is to put the movable zero nearly equal to the attenuation factor of the complex poles.

The general discussion about using low order signals to compensate high order systems will be given in Chapter VI.

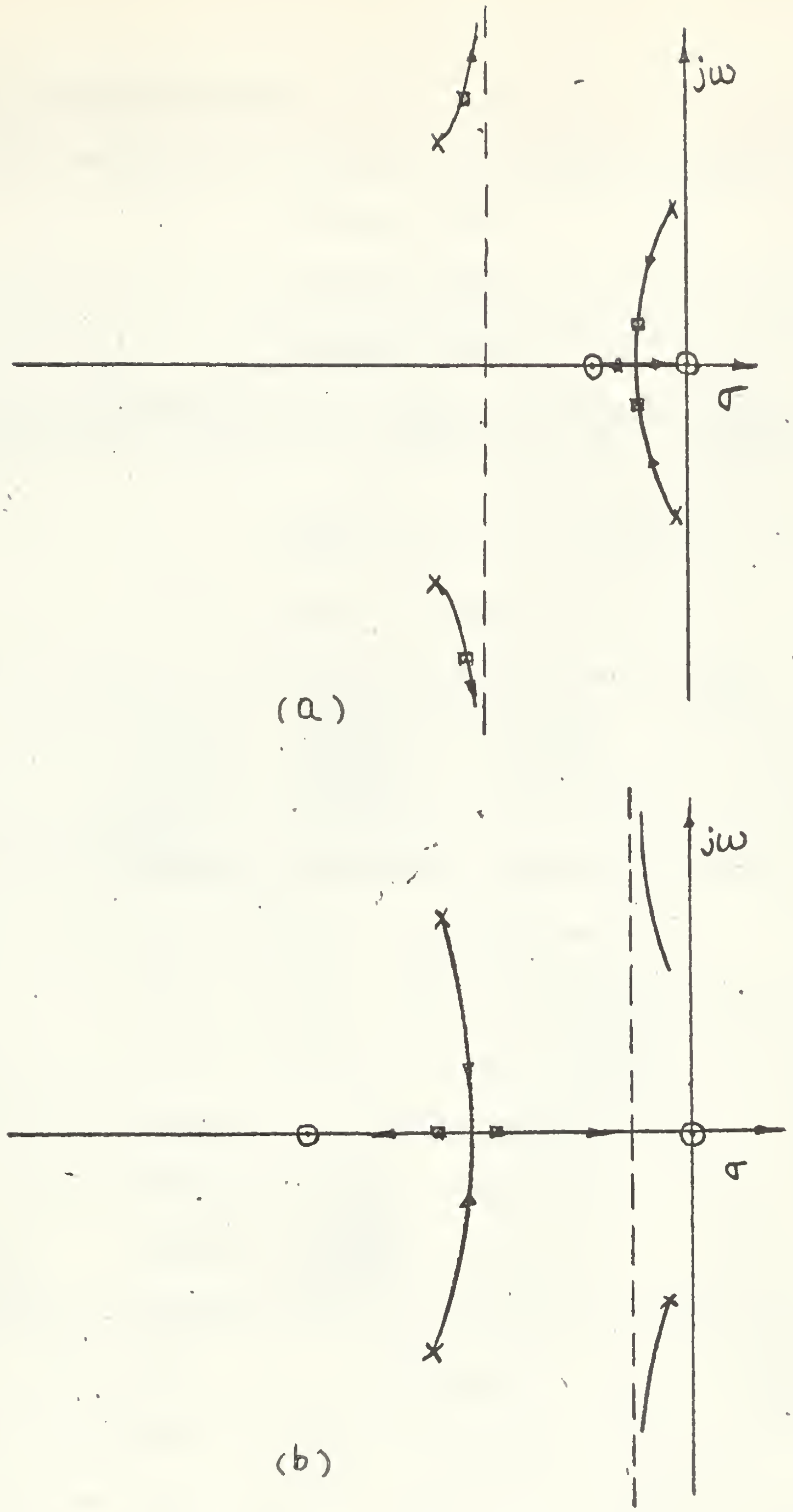


Fig. 2-37 Possible Results of Compensation by Feeding Back θ_c and $\dot{\theta}_c$.
(a) Zero too close to the origin
(b) Zero far away from the origin

(C) Feed back determination:

From the relative max. values of θ_o, $\dot{\theta}_o$ & $\ddot{\theta}_o$ in Fig. 2-38

$$\theta_{o\,max.} = 5.95^V$$

$$\dot{\theta}_{o\,max.} = 3.05^V$$

$$\ddot{\theta}_{o\,max.} = 0.16^V$$

Since for sine wave output $\theta_{o\,max.} \simeq A = 5.95^V$ The max. values of $\dot{\theta}_o$ & $\ddot{\theta}_o$ are:

$$\dot{\theta}_{o\,max.} = \omega_n A \qquad \omega_n = 8$$

$$\ddot{\theta}_{o\,max.} = \omega_n^2 A$$

Then the attenuations of the output of $\dot{\theta}_o$ & $\ddot{\theta}_o$ are:

$$\alpha_{\dot{\theta}_o} = \frac{3.06}{5.95 \times 8} = 0.064$$

$$\alpha_{\ddot{\theta}_o} = \frac{0.16}{5.95 \times 64} = 0.00042$$

From the over damped characteristic equation for roots at -3, -6, and -11.5 $\pm$ j33.7, the $\dot{\theta}_o$ & $\ddot{\theta}_o$ feedbacks are:

$$Kh = 9750, \quad h = \frac{9750}{23774.29} = 0.4105$$

$$km = 1038, \quad m = \frac{1038}{23774.29} = 0.0435$$

Then the amplification of $\dot{\theta}_o$ & $\ddot{\theta}_o$ should be:

$$A_{\dot{\theta}_o} = \frac{0.4105}{0.064} = 6.41$$

$$A_{\ddot{\theta}_o} = \frac{0.0435}{0.00042} = 103.9$$

But from the amplifier circuit for $(\theta_i - \theta_o)$, when $a_{\theta_o} = a_{\theta_i} = 0.1179$ and $R_{\theta_i} = R_{\theta_o} = 1$ M, the gain is $W_x = \frac{a_{\theta_i} R_f}{R_{\theta_i}} = 2.7$.

This amplification of $(\theta_i - \theta_o)$ is already included in the gains of the open loop transfer function. Then the total gain for the feedback signals when used this same amplifier must be

$$A_{\dot{\theta}_o} \times 2.7 = 17.3$$

$$A_{\ddot{\theta}_o} \times 2.7 = 280$$

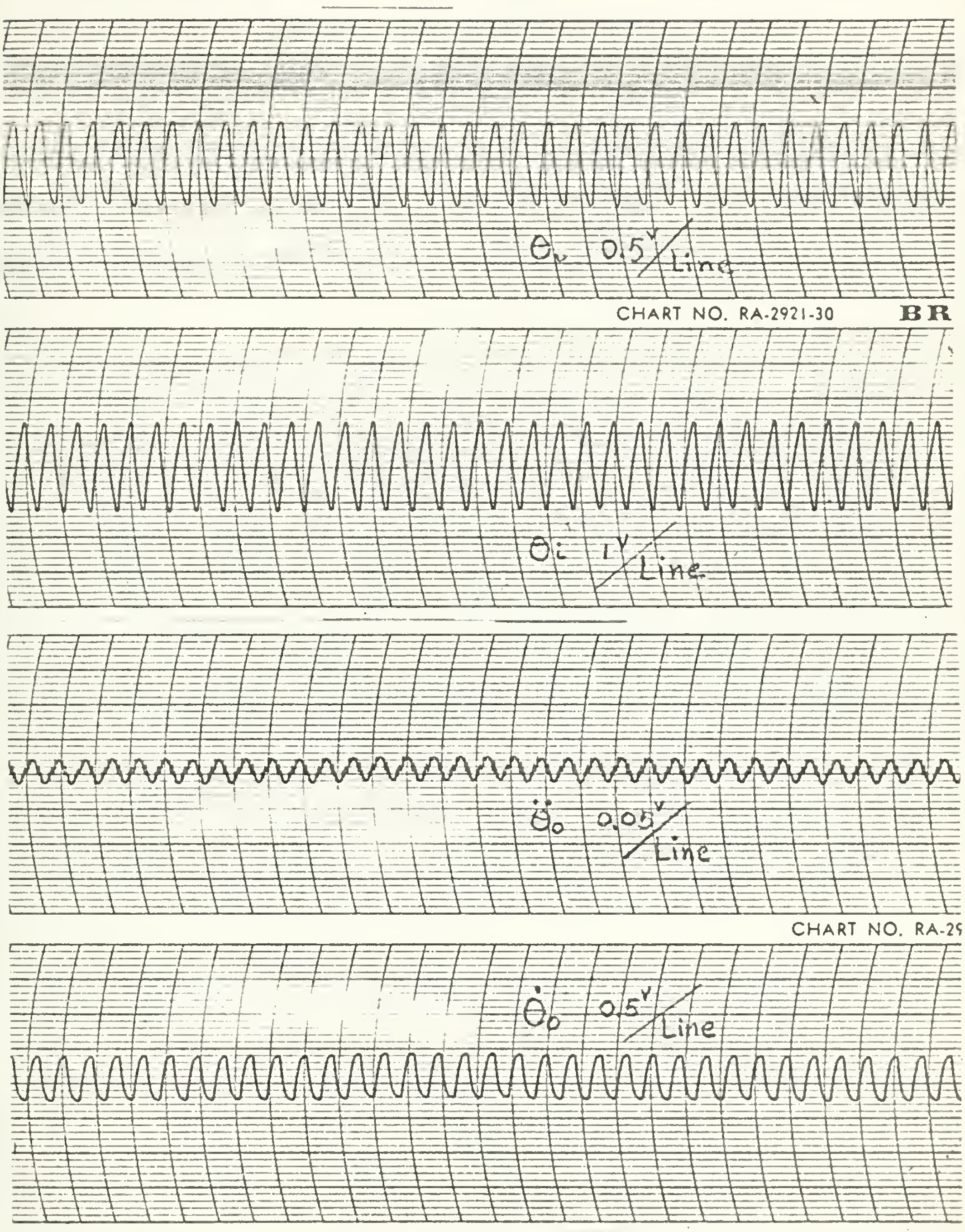


Fig. 2-38 Sine Waves Test for Feedback Determination

For $R_{\dot{\theta}_0} = 0.5$ M, $R_f = 23$M

$R_{\ddot{\theta}_0} = 0.05$M "

The values of $a_{\dot{\theta}_0}$ and $a_{\ddot{\theta}_0}$ are:

$$a_{\dot{\theta}_0} = \frac{17.3 \times 0.5}{23} = 0.375$$

$$a_{\ddot{\theta}_0} = \frac{280 \times 0.05}{23} = 0.61$$

By using $h = 0.4105$, $m = 0.0435$ the root locus of the overdamped characteristic equation is plotted in Fig. 2-39, and the tested characters are indicated in Fig. 40, 41.

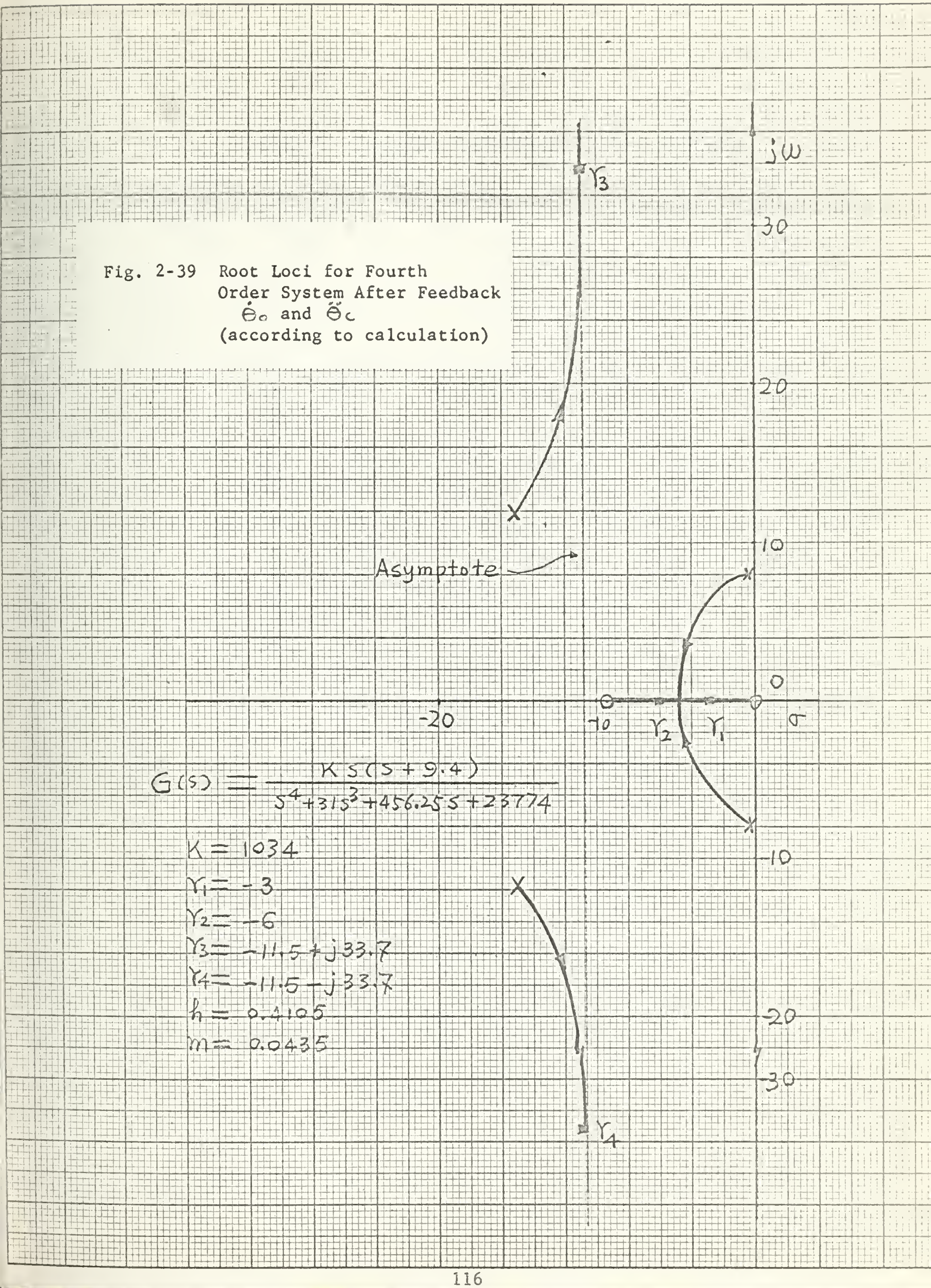


Fig. 2-39 Root Loci for Fourth Order System After Feedback $\dot{\theta}_o$ and $\ddot{\theta}_c$ (according to calculation)

TABLE 2-7 Over Damped System Open Loop Frequency Response.

f cyc/sec.	ω rad/sec	θ_0/θ_i	db
0.01	0.0628	1	0
0.12	0.75	0.99	-0.087
0.125	0.782	0.984	-0.14
0.5	3.13	0.966	-0.3
0.6	3.76	0.958	-0.372
0.7	4.39	0.931	-0.62
0.8	5.01	0.902	-0.894
0.9	5.64	0.897	-0.941
1.0	6.28	0.888	-1.028
1.2	7.52	0.881	-1.1
1.5	9.42	0.867	-1.24
2.0	12.56	0.857	-1.34
2.5	15.7	0.818	-1.74
3.0	18.84	0.892	-2.025
4.0	25.1	0.782	-2.13
4.5	28.25	0.68	-3.34
5.0	31.4	0.496	-6.08
5.5	34.55	0.397	-8.0
6.0	37.6	0.322	-9.21
6.5	40.8	0.2425	-12.2
7.0	44	0.186	-14.58
7.5	47	0.1738	-15.2
8.0	50.1	0.1488	-16.5
8.5	53.2	0.124	-18.1
9.0	56.5	0.099	-20
10.0	62.8	0.099	-20

Output taken from Tachometer

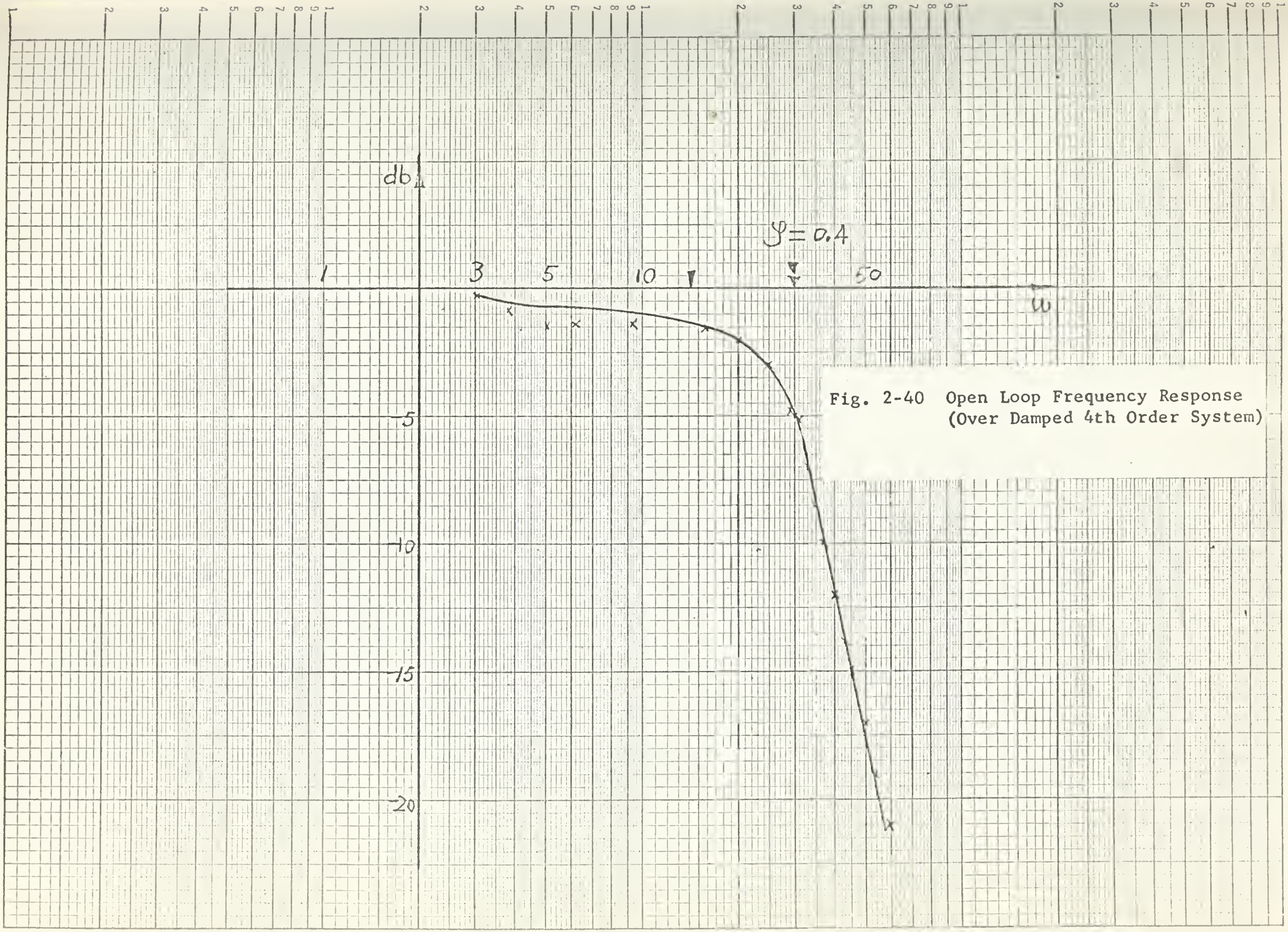


Fig. 2-40 Open Loop Frequency Response
(Over Damped 4th Order System)

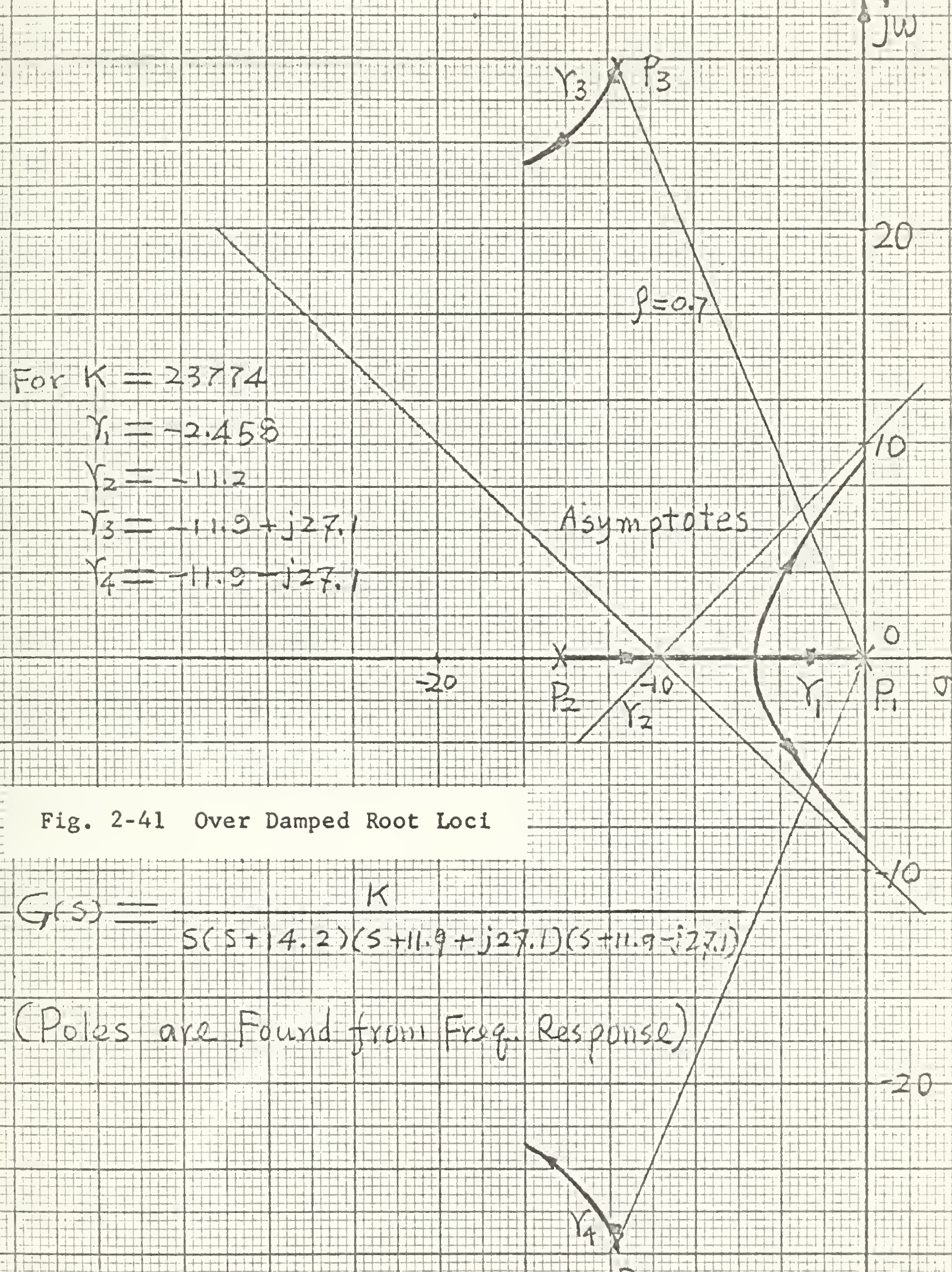


Fig. 2-41 Over Damped Root Loci

$$G(s) = \frac{K}{s(s+14.2)(s+11.9+j27.1)(s+11.9-j27.1)}$$

(Poles are Found from Freq. Response)

(E) Calculations for Control Signal

By using positive value of roots

$$\gamma_1 = +3$$

$$\gamma_2 = +6$$

$$\gamma_3 = +11.5 + j33.7$$

$$\gamma_4 = +11.5 - j33.7$$

$$u = \gamma_2\gamma_3\gamma_4 x + (\gamma_2\gamma_3 + \gamma_2\gamma_4 + \gamma_3\gamma_4)\dot{x} + (\gamma_2 + \gamma_3 + \gamma_4)\ddot{x} + \dddot{x} = 0$$

or $$u = (S+6)(S+11.5+j33.7)(S+11.5-j33.7)$$

$$= (S+6)(S^2 + 23.5 + 12.67)$$

$$= S^3 + 29S^2 + 1345S + 7602 = 0$$

i.e. $$7602x + 1345\dot{x} + 29\ddot{x} + \dddot{x} = 0 \qquad (2\text{-}69)$$

If only x and $\dot{x}$ are used to produce the control signal then

$$J = 7602x + 1345\dot{x} = 0$$

$$-\dot{\theta}_o = \dot{x} = -\frac{7602}{1345}x \qquad \text{(For step input)}$$

$$\dot{\theta}_c = 5.66x = kx$$

Since the term $\ddot{x}$ is near maximum value at the switching time, a correction is needed.

From the computer x-y plotter at switching time

$x = 0.6$	rad.
$\dot{\theta}_o = 6.6$	rad/sec.
$\ddot{\theta}_o = 60$	rad/sec.^{2}
$\dddot{\theta}_c = 51.2$	rad/sec.^{3}

Then a correction of k is $\frac{29\ddot{\theta}_o}{1345\dot{\theta}_c} = \frac{29\times60}{1345\times6.6} = \frac{1740}{89000} = 0.196$

This means k has to be 5.66 x (1-0.196) = 4.55. Here the effect of $\dddot{\theta}_c$ can be neglected.

(F) Calculations for Setting the Parameters in the Control Signal Amplifier Circuit.

From the above calculation by using χ and $\dot{\theta}_o$ to produce the control signal the relation of χ to $\dot{\theta}_o$ is:

$$\dot{\theta}_o = 4.55\,\chi$$

$$\text{or} \qquad \chi = 2.220\,\dot{\theta}_o$$

In the actual setting $R_{fJ} = 1M$, $R_{J\theta_i} = R_{J\theta_o} = 0.5M$, and $a_{J\theta_i} = a_{J\theta_o} = 0.118$. The amplification is:

$$A_{\theta_i} = A_{\theta_o} = \frac{0.118 \times 1}{0.5} = 0.236$$

then the amplitude of $\dot{\theta}_o$ has to be $0.22 \times 0.236 = 0.052$.

Since the attenuation of $\dot{\theta}_o$ (from previous test) is 0.064, and $\omega_n = 8$.

Then the total amplification of this control signal amplifier has to be

$$a_{J\dot{\theta}_o} = \frac{0.052}{0.064} = 0.813$$

For $R_{fJ} = 1M$, $R_{J\dot{\theta}_o} = 0.1M$, then $a_{J\dot{\theta}_o} = \frac{0.813 \times 0.1}{1} = 0.0813$

The results from the above calculations are indicated in Fig. 2-42. Since the system operated at these setting values is not critical, the value of $a_{J\dot{\theta}_o}$ can be used from 0.831 to 0.98. (See Fig. 2-13b for different switching time).

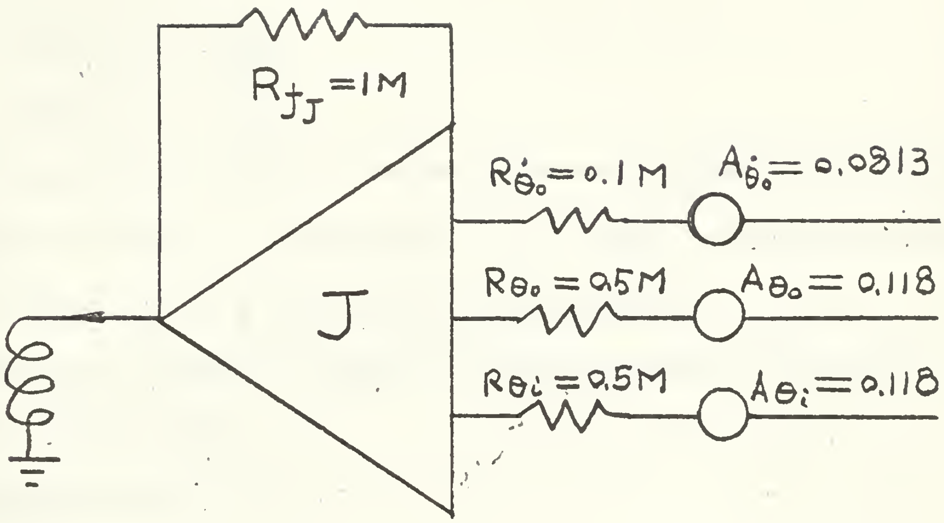


Fig. 2-42 The setting Values for Control Signal Amplifier

(III) Design of R-C Differentiator for Producing $\dot{x}$ and $\ddot{x}$ signals

Since the R-C differentiator output signal cannot be accurately calculated, using the x-y plotter to find the attenuation and phase shift is a practical method.

By using $R = 0.5M$, $C = 0.1\mu f$ in Fig. 2-43.

The attenuation of $\dot{x}$ and $\ddot{x}$ are:

$$\alpha_{\dot{x}} = \frac{1}{267}$$

$$\alpha_{\ddot{x}} = \frac{1}{267}$$

Then by amplifying $\dot{x}$ and $\ddot{x}$ as many times according to the attenuation or reducing the x signal (that feeds to J amplifier) proportionally, the control signal can be produced without much error.

A comparison of these two signals to those produced the computer amplifier is in Fig. 2-45.

According to Fig. 2-44,

$$\frac{u}{100} = 0.0774\,x + 0.0587 \times 2.67\,\dot{x} + 0.01 \times 2.67\,\ddot{x}$$
$$= W_x x + W_{\dot{x}} \bar{\dot{x}} + W_{\ddot{x}} \bar{\ddot{x}}$$

Amplification needed $A_x = 0.25$

$A_{\dot{x}} = 71.7$

$A_{\ddot{x}} = 66.8$

$W_x = W_{11} = 0.25 \times 1.16 = 0.29$	$R_{11} = 1$	$a_{11} = 0.029$
$W_{\dot{x}} = W_{12} = 71 \times 0.587 = 41.6$	$R_{12} = 0.1$	$a_{12} = 0.41$
$W_{\ddot{x}} = W_{10} = 66.8 \times 0.05 = 3.34$	$R_{10} = 1$	$a_{10} = 0.334$

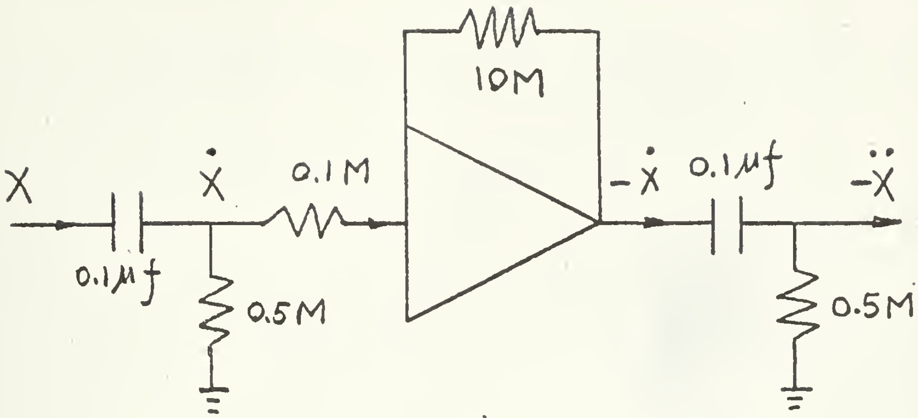


Fig. 2-43 Differentiating Circuit by using R-C Differentiator and Amplifier

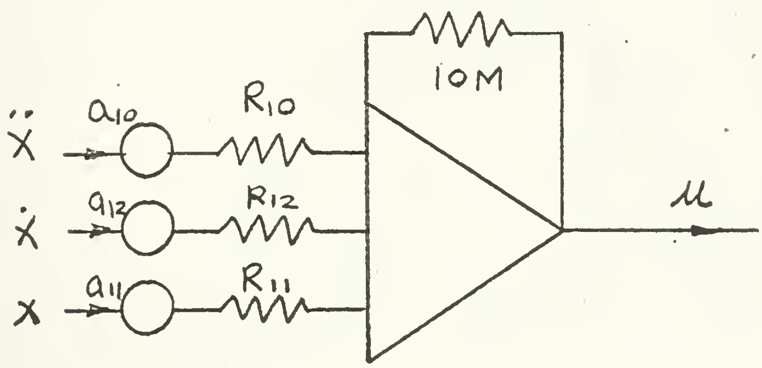


Fig. 2-44 Circuit Diagram of Control Signal Amplifier for Feeding back θ_o, $\dot{\theta}_o$ and $\ddot{\theta}_o$ signals.

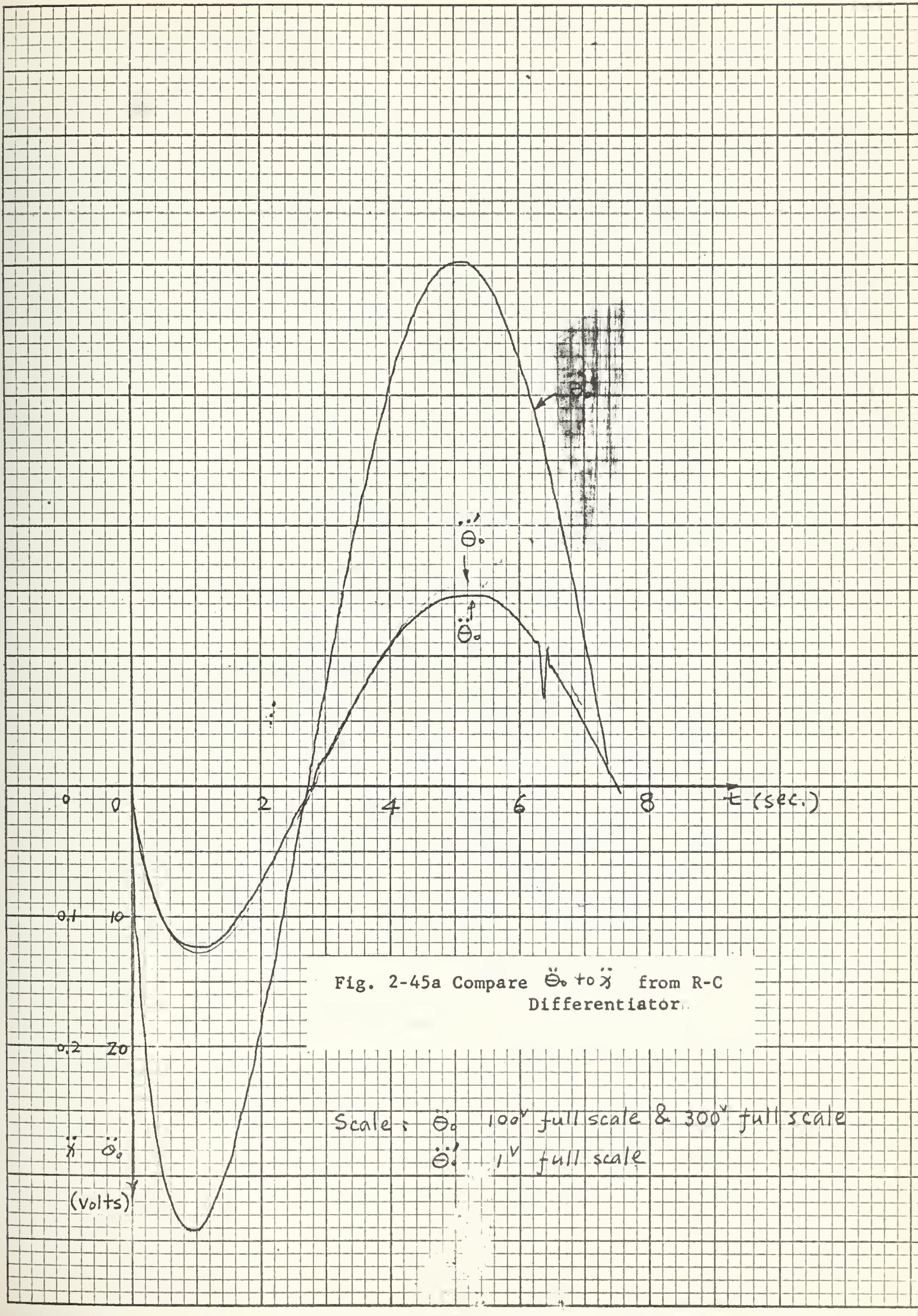


Fig. 2-45a Compare $\ddot{\theta}_o$ to $\ddot{x}$ from R-C Differentiator

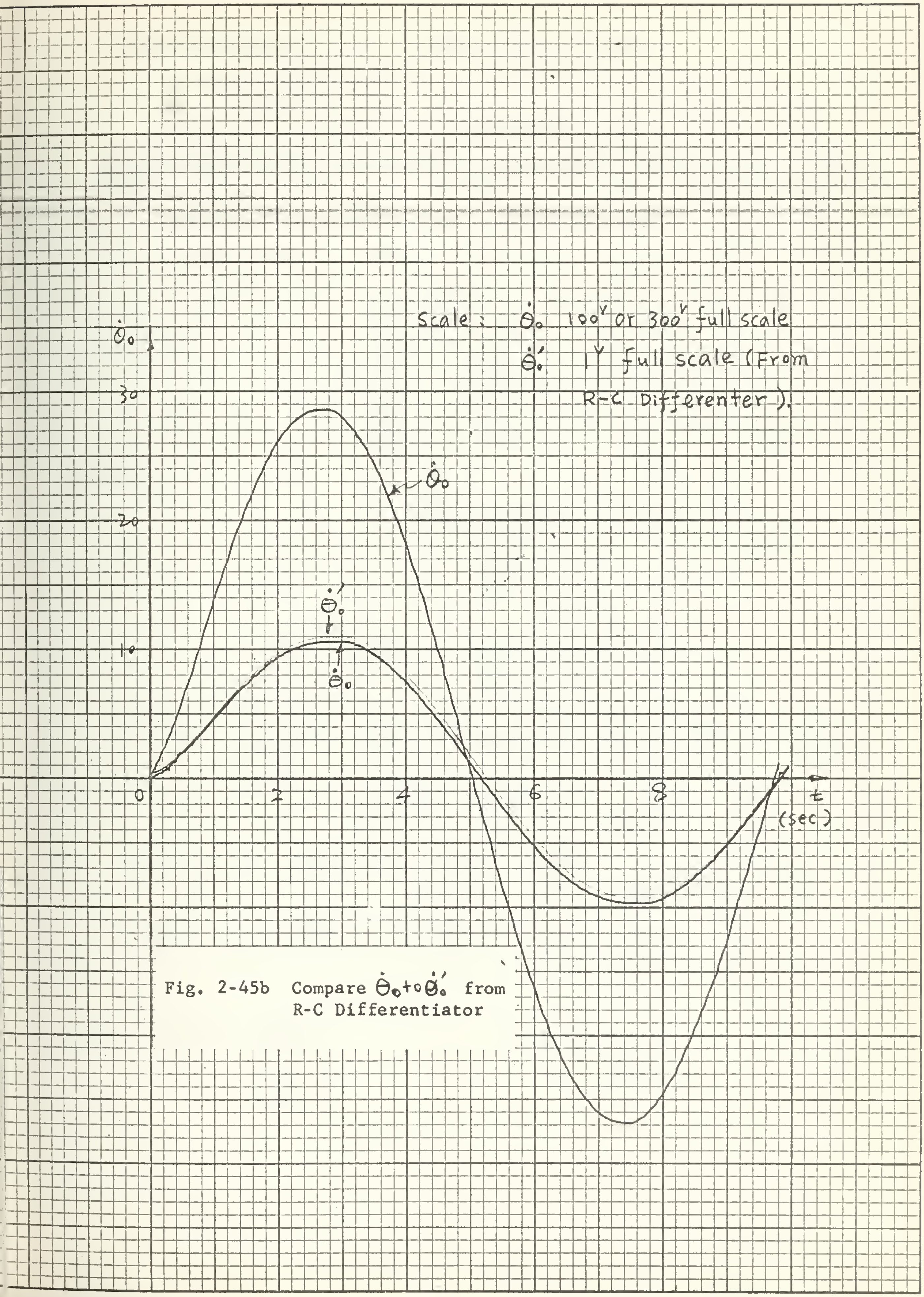


Fig. 2-45b Compare $\dot{\Theta}_o$ to $\dot{\Theta}_o'$ from R-C Differentiator

(G) Analog Computer Study for Fourth Order System

(1) Underdamped case:

Characteristic equation for underdamped case is

$$D_{F_{c1}}(S) = D^4\bar{\Theta}_o + 31D^3\bar{\Theta}_o + 456.25D^2\bar{\Theta}_o + 2178D\bar{\Theta}_o + 23774.29\Theta_o \qquad (2\text{-}70)$$

For unit step input with zero initial conditions

$$X_{max.} = \Theta_{i\,max.} = \Theta_{o\,max.} = 1$$

$$\dot{\Theta}_{o\,max.} = 8$$

$$\ddot{\Theta}_{o\,max.} = 64$$

$$\dddot{\Theta}_{o\,max.} = 512$$

$$\ddddot{\Theta}_{o\,max.} = 4100$$

By using

$$\alpha_X = \alpha_{\Theta_i} = \alpha_{\Theta_o} = 0.0333$$

$$\alpha_{\dot{\Theta}_o} = 0.26$$

$$\alpha_{\ddot{\Theta}_o} = 2.1$$

$$\alpha_{\dddot{\Theta}_o} = 16.66$$

$$\alpha_{\ddddot{\Theta}_o} = 543$$

$$-\overline{D^4\Theta_o} = \frac{1}{\alpha_{\ddddot{\Theta}_o}}\left(23774.29\alpha_X\overline{X} - 31\alpha_{\dddot{\Theta}_o}\overline{D^3\Theta_o} - 456.25\alpha_{\ddot{\Theta}_o}\overline{D^2\Theta_o} - 2178\alpha_{\dot{\Theta}_o}\overline{D\Theta_o}\right)$$

$$= W_{10}\overline{X} - W_2\overline{D^3\Theta_o} - W_3\overline{D^2\Theta_o} - W_4\overline{D\Theta_o}$$

$$\overline{D^3\Theta_o} = \frac{1}{P}(-\overline{D^4\Theta_o}) = W_6\overline{D^4\Theta_o}, \qquad -\overline{D^2\Theta_o} = \frac{1}{P}(+\overline{D^3\Theta_o}) = W_7\overline{D^3\Theta_o}$$

$$\overline{D\Theta_o} = \frac{1}{P}(-\overline{D^2\Theta_o}) = W_8\overline{D^2\Theta_o}, \qquad -\overline{\Theta_o} = \frac{1}{P}(\overline{D\Theta_o}) = W_9\overline{D\Theta_o}$$

where $W_n = \dfrac{a_n R_{fn}}{R_n}$ for summers

$W_n = \dfrac{a_n}{R_n C_{fn}\alpha_t}$ for integrators

(II) Overdamped case:

From Fig. 2-41 the overdamped characteristic equation is:

$$D_{Fc_2}(S) = S^4 + 31S^3 + 1494.25S^2 + 11938S + 23774.29 \tag{2-71}$$

From Eq. (2-70) we have:

$$D_{Fc_2}(S) - D_{Fc_1}(S) = 1038S^2 + 9750S = W_3' \overline{D^3\Theta_o} - W_4' \overline{D^2\Theta_o}$$

Then adding this feedback $\overline{D^2\Theta_o}$ and $\overline{D'\Theta_o}$ the system will be overdamped.

(III) Computer setup and recording results: ----

The block diagram for computer setup of this 4th order system is in Fig. 2-46. The setting values of each component is in Table 2-8. The comparison of using θ_o and $\dot{\theta}_o$ switching with that by using θ_o, $\dot{\theta}_o$ $\ddot{\theta}_o$ and $\dddot{\theta}_o$ switching is recorded in Fig. 2-47. The step response character is indicated in Fig. 2-48. The effect of switching time is plotted in Fig. 2-49. The changing of the various derivative signals with respect to time is given in Fig. 2-50. And a phase space model can be made by using Fig. 2-51, and Fig. 2-52. (A simplified model must be used to reduce the system order to three in order to construct the trajectories in the conventional three dimension phase space.)

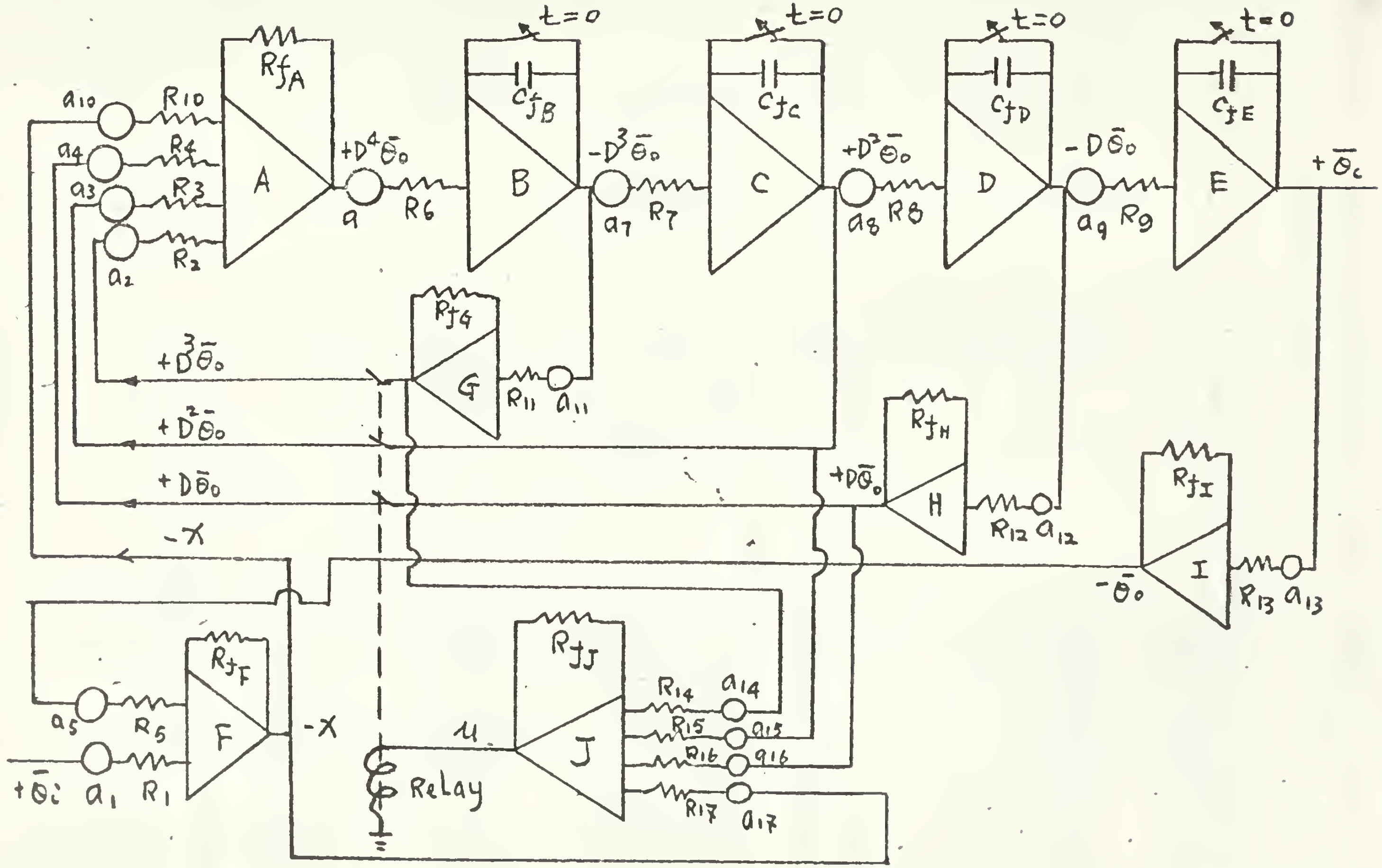


Fig. 2-46 Computer setup for 4th Order System

TABLE 2-8 Computer Setting Values For a Fourth Order Physical System

$W_1 = 1$	$R_{tf} = 1$	$R_1 = 0.51$	$a_1 = 0.51$
$W_5 = 1$	$R_{tf} = 1$	$R_5 = 0.507$	$a_5 = 0.507$
$W_{10} = 1.485$	$R_{fA} = 10$	$R_{10} = 1.015$	$a_{10} = 0.151$
$W_2 = 0.968$		$R_2 = 1.01$	$a_2 = 0.0978$
$W_3 = 1.79$		$R_3 = 1.00$	$a_3 = 0.179$
$W_4 = 1.06$		$R_4 = 1.00$	$a_4 = 0.106$
$W_3 = 4.07$	$R_{fA} = 10$	$R_3 = 0.1$	$a_3 = 0.0407$
$W_4 = 4.75$		$R_4 = 0.105$	$a_4 = 0.05$
$W_6 = 3.205$	$C_{fB} = 1.0$	$R_6 = 0.106$	$a_6 = 0.34$
$W_7 = 0.7933$	$C_{fC} = 1.06$	$R_7 = 0.502$	$a_7 = 0.4221$
$W_8 = 0.8076$	$C_{fD} = 1.06$	$R_8 = 0.51$	$a_8 = 0.436$
$W_9 = 0.78$	$C_{fE} = 1.05$	$R_9 = 0.5$	$a_9 = 0.41$
$W_{11} = 1$	$R_{fG} = 1$	$R_{11} = 0.509$	$a_{11} = 0.509$
$W_{12} = 1$	$R_{fH} = 1$	$R_{12} = 0.505$	$a_{12} = 0.505$
$W_{13} = 1$	$R_{fI} = 1$	$R_{13} = 0.509$	$a_{13} = 0.509$
$W_{16} = 8.14$	$R_{fJ} = 2$	$R_{16} = 1.015$	$a_{16} = 0.885$
$W_{17} = 9.5$		$R_{17} = 0.505$	$a_{17} = 0.2525$
		change to 0.285 for getting better response	
$W_{14} = 0.0674$	$R_{fJ} = 2$	$R_{14} = 1$	$a_{14} = 0.0337$
$W_{15} = 0.234$		$R_{16} = 1$	$a_{16} = 0.1173$

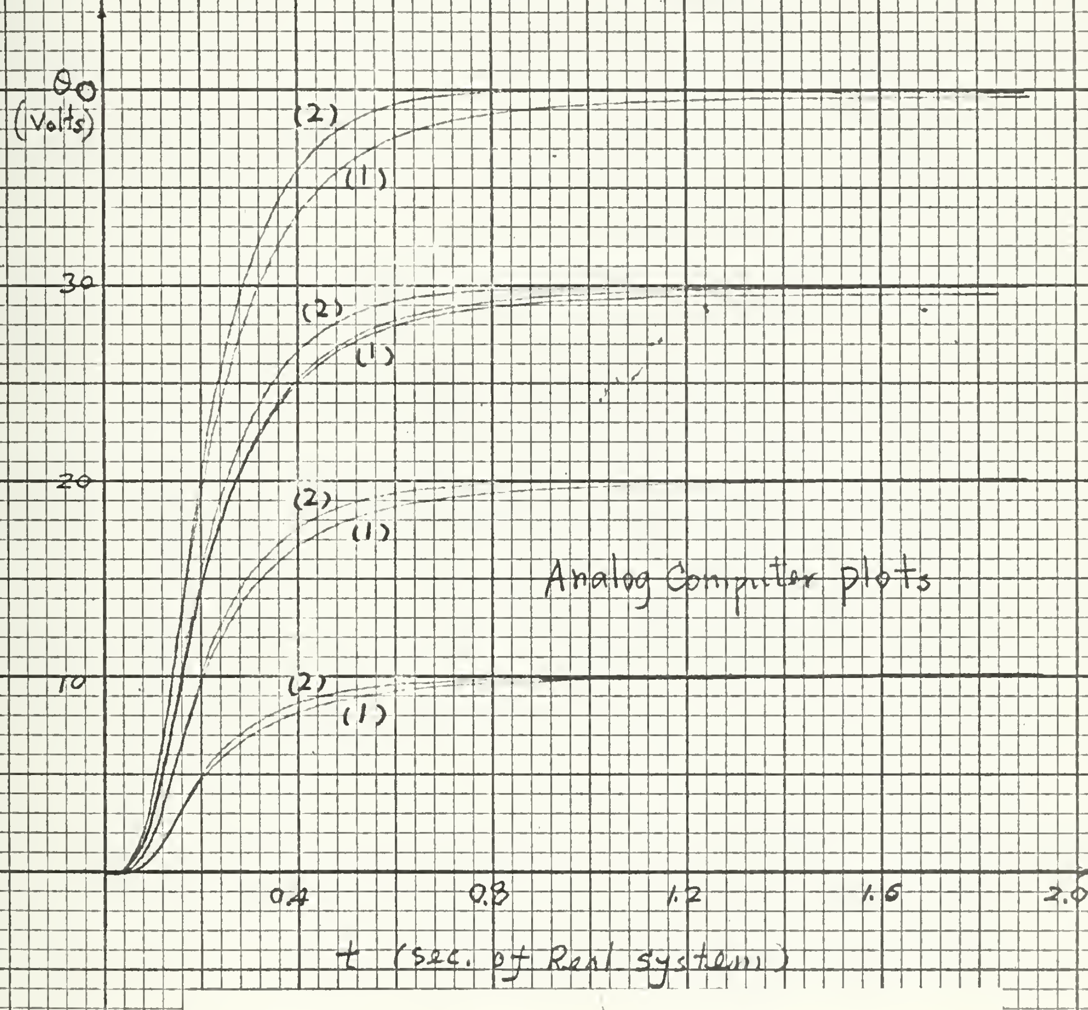


Fig. 2-47 A Comparison of Step Response
(1) Switched by $\dddot{\theta}_o$, $\ddot{\theta}_o$, $\dot{\theta}_o$ and θ_o
(2) Switched by $\dot{\theta}_o$ and θ_o

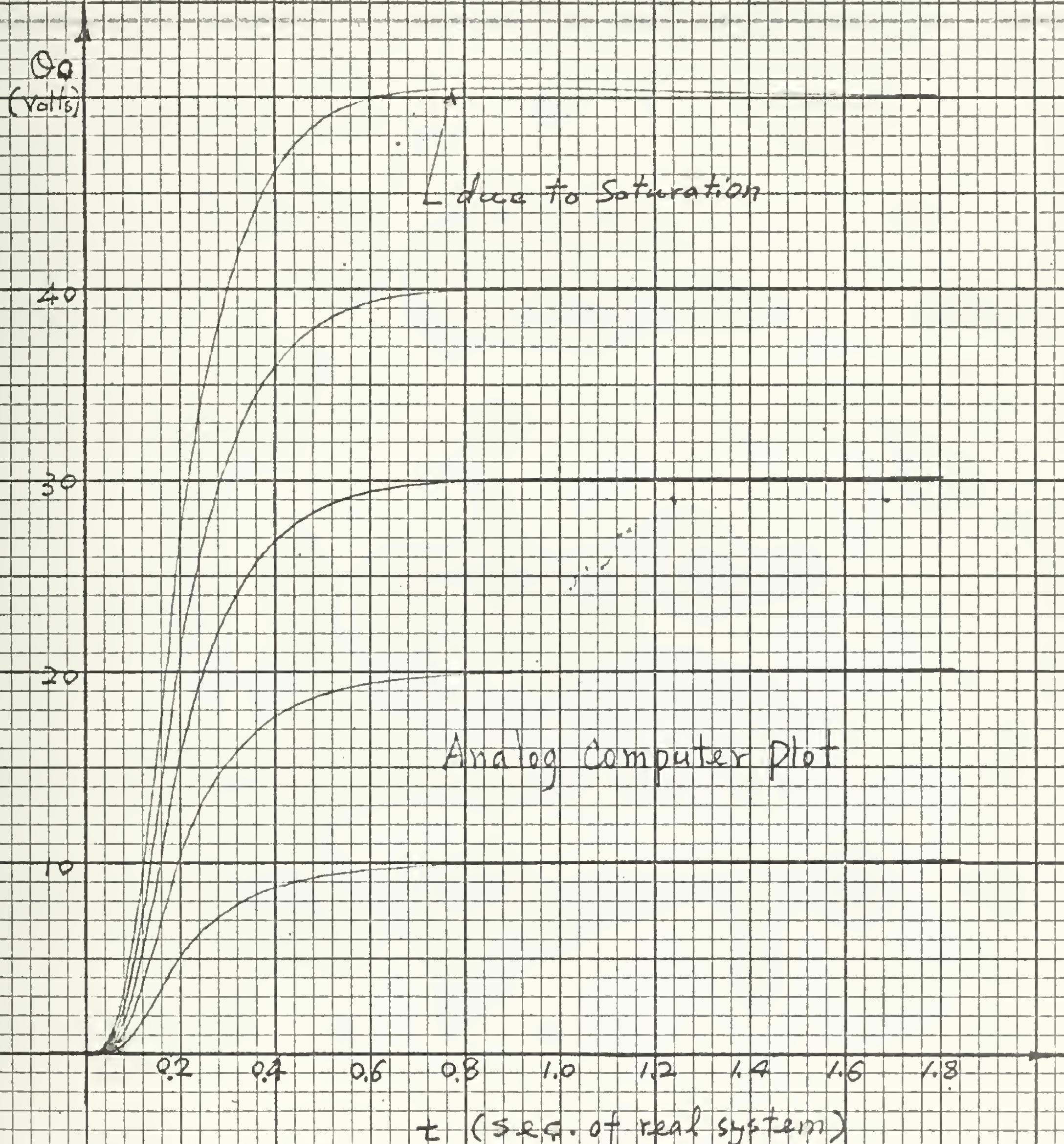


Fig. 2-48 Step Response of a Fourth Order Servo System. (From Analog Computer)

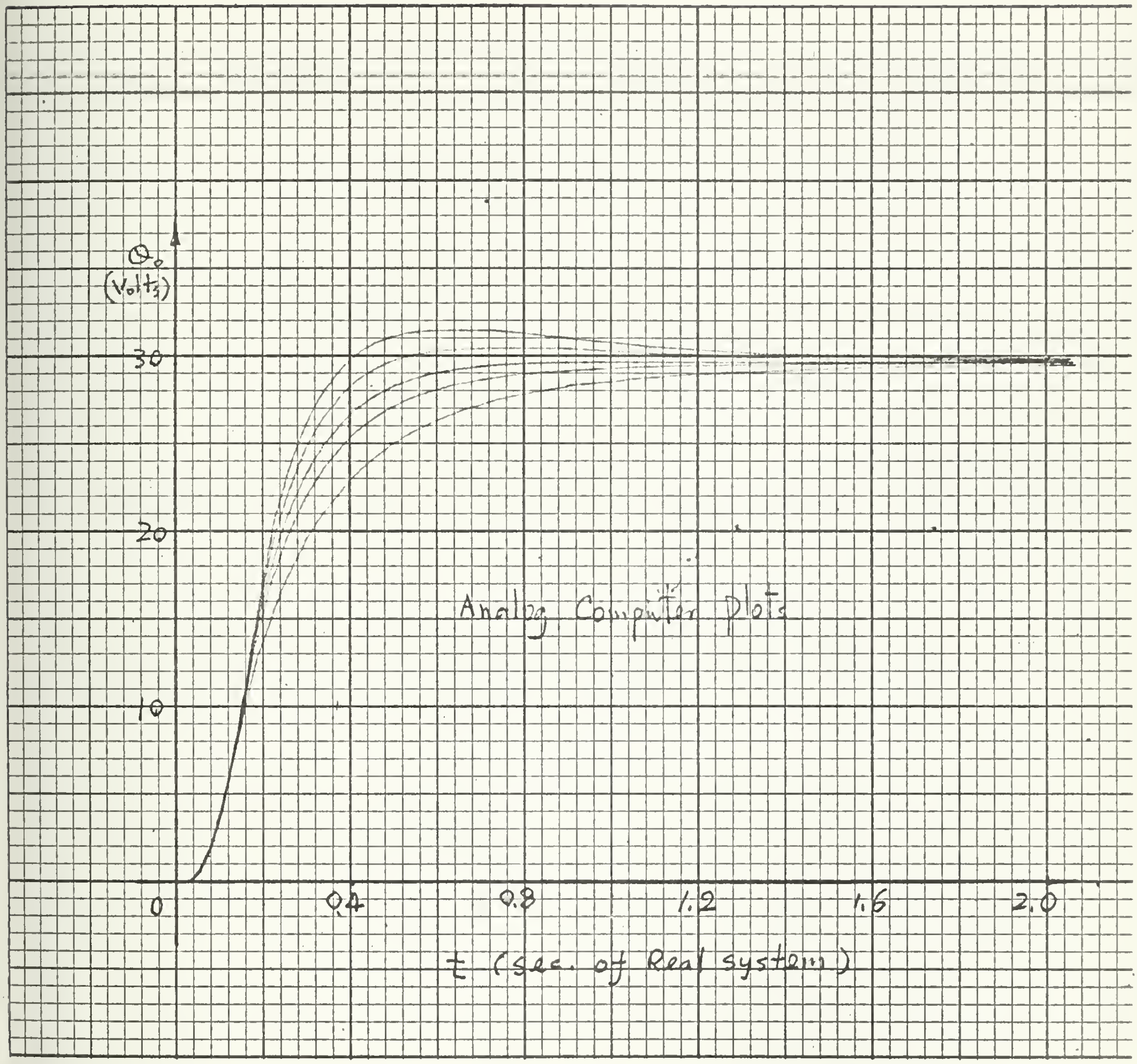


Fig. 2-49 The Effect of Switching Time on a Fourth Order Servo System.

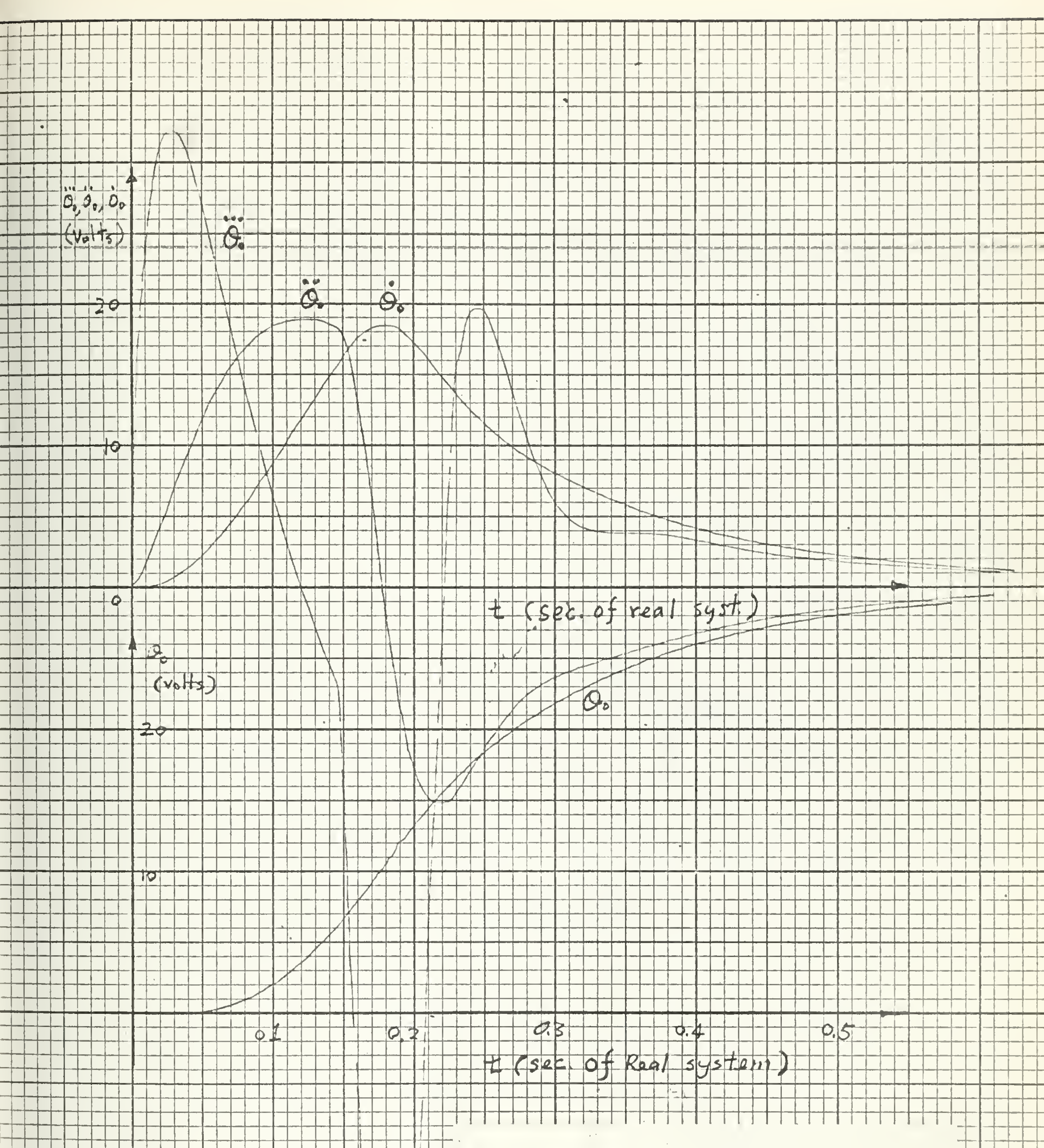


Fig. 2-50 Analog Computer Plots for a Fourth Order Servo with Discontinuous Damping

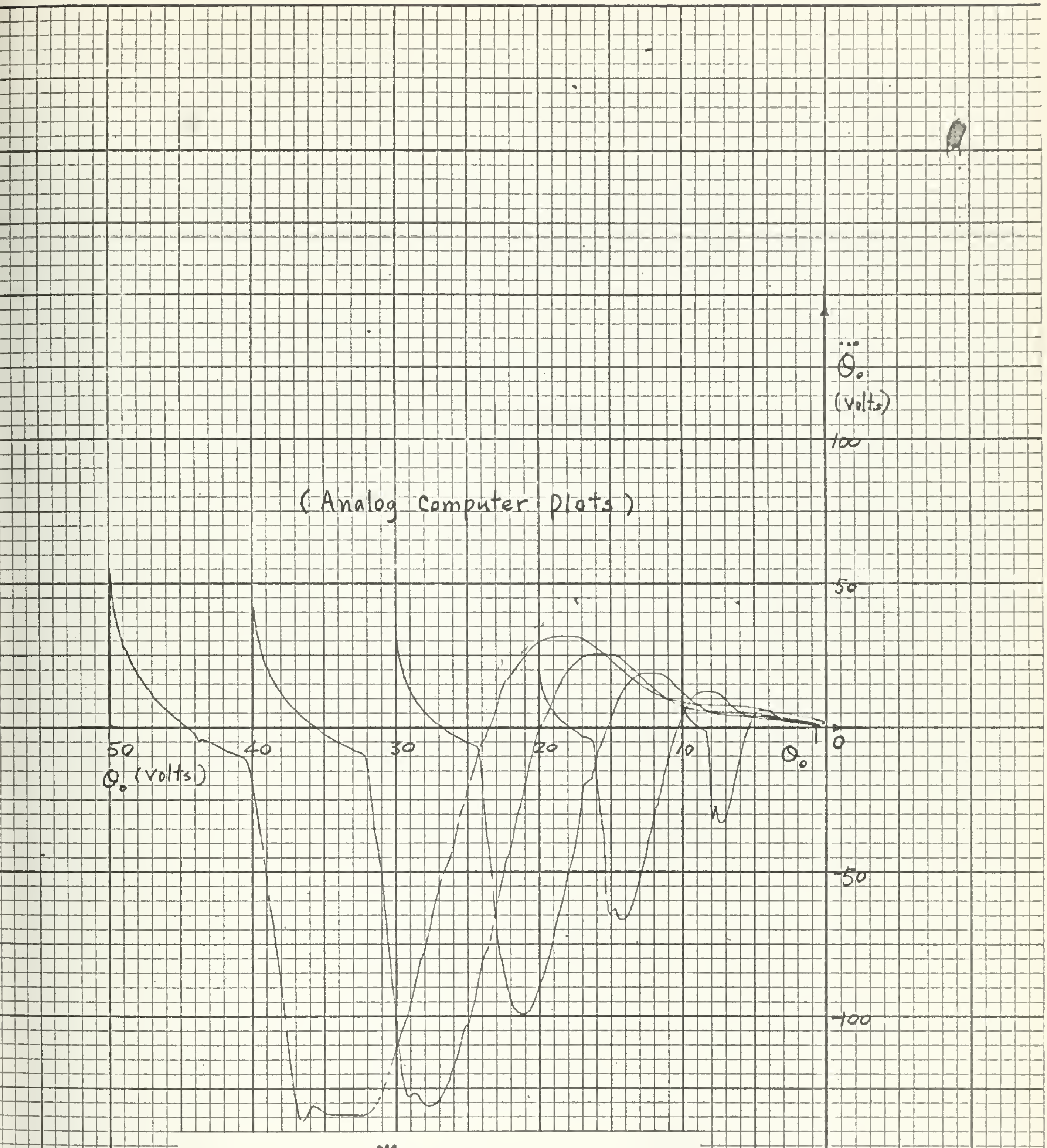


Fig. 2-51 $\dddot{\theta}_o$ vs. θ_o For Step Input For 4th Order System

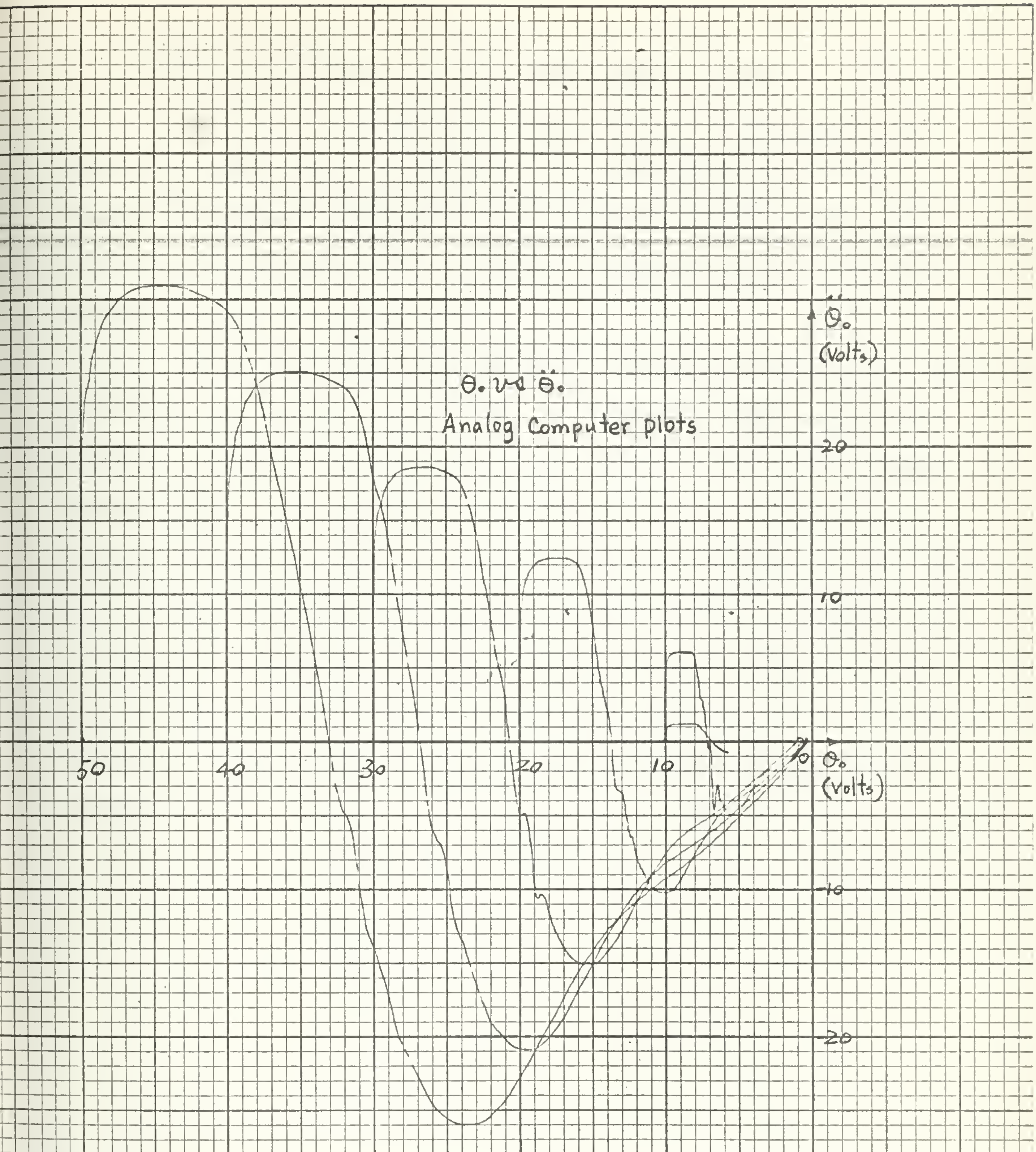


Fig. 2-52 Step Response Plots for
4th Order System

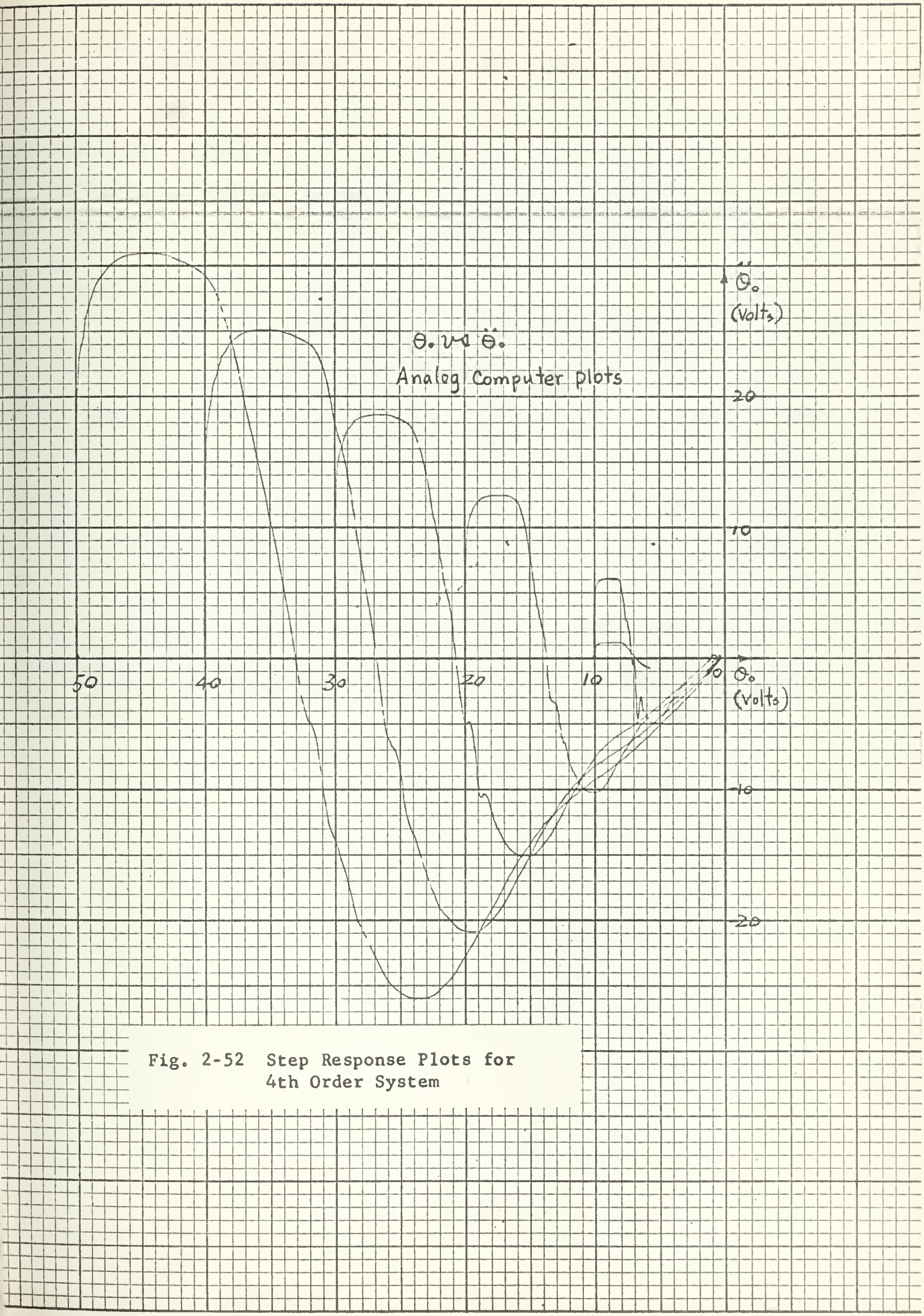


Fig. 2-52 Step Response Plots for
4th Order System

(H) The Tested Results of a 4th Order Physical System.

A picture of the tested 4th order system is in Fig. 2-53. The motor-load combination is at the upper left side and the amplifier is below the motor-load combination. The amplidyne is at the center. The control signal circuits are at right side of the picture, they are taken off from the analog computer.

The various signals for step input are plotted in Fig. 2-54. From this figure can be seen how noisy the $\ddot{\theta}_o$ signal is. The ramp response curves are plotted in Fig. 2-55, the output curves of θ_o and $\dot{\theta}_o$ indicate that the system is at the chattering condition, this will be discussed in Chapter V. Fig. 2-56 shows the slow response of the system to a step input due to too small $\ddot{\theta}_o$ feedback, and Fig. 2-57 shows the slow response due to too large $\ddot{\theta}_o$ feedback. If the feedback of $\ddot{\theta}_o$ is increased further, then oscillation will occur as recorded in Fig. 2-58. The slow response and oscillation characters are discussed in (B) of this section and also in Chapter VI.

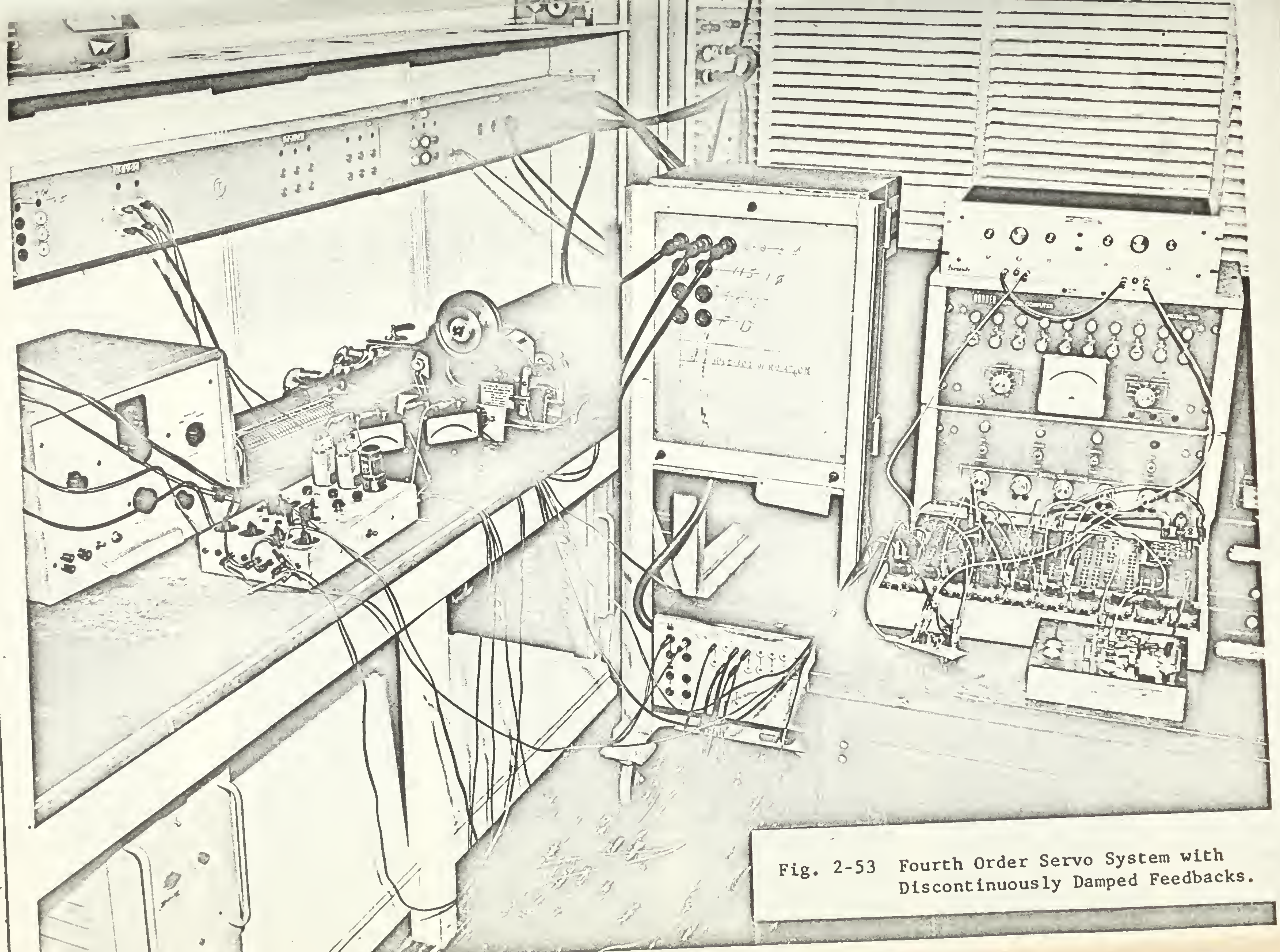


Fig. 2-53 Fourth Order Servo System with Discontinuously Damped Feedbacks.

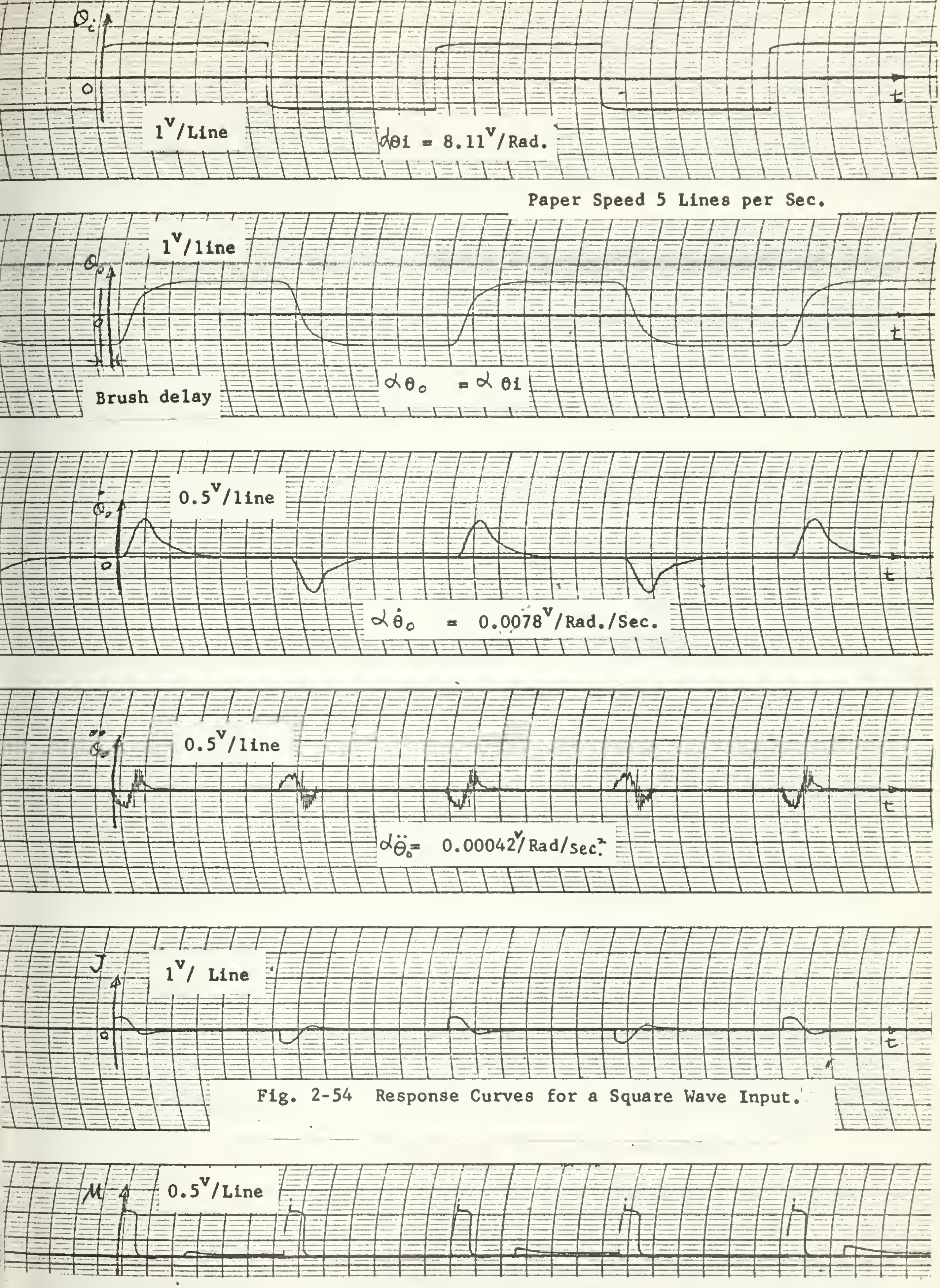


Fig. 2-54 Response Curves for a Square Wave Input.

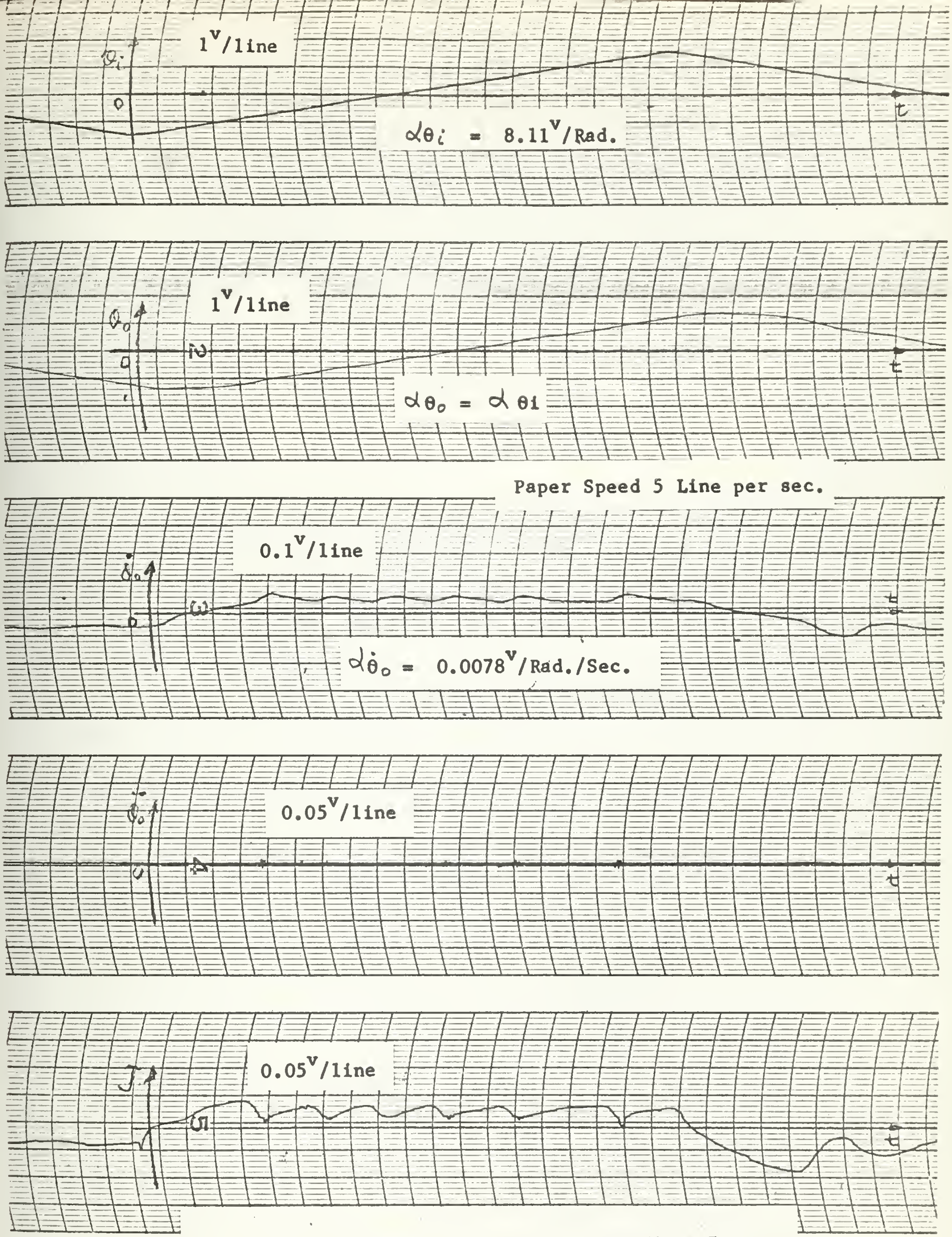


Fig. 2-55 Response Curves for a Ramp Wave Input.

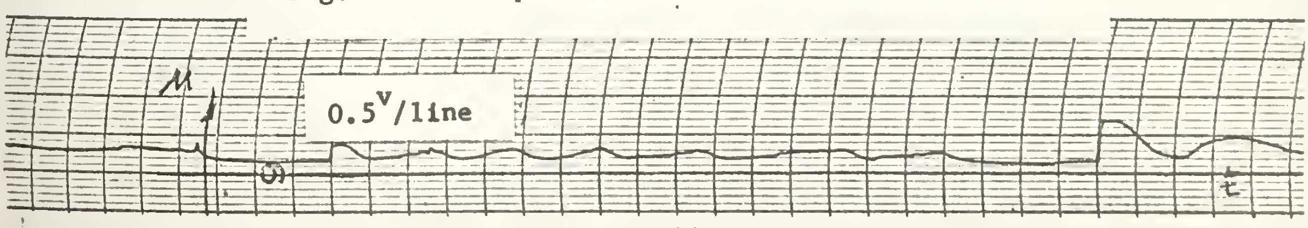


Fig. 2-56 Slow Response to A Step Input due to too Small $\ddot{\theta}_c$ Feedback.

$a_{\ddot{\theta}_c} = 0.65 \quad R_{\ddot{\theta}_c} = 0.1M\,\Omega$

(Real System)

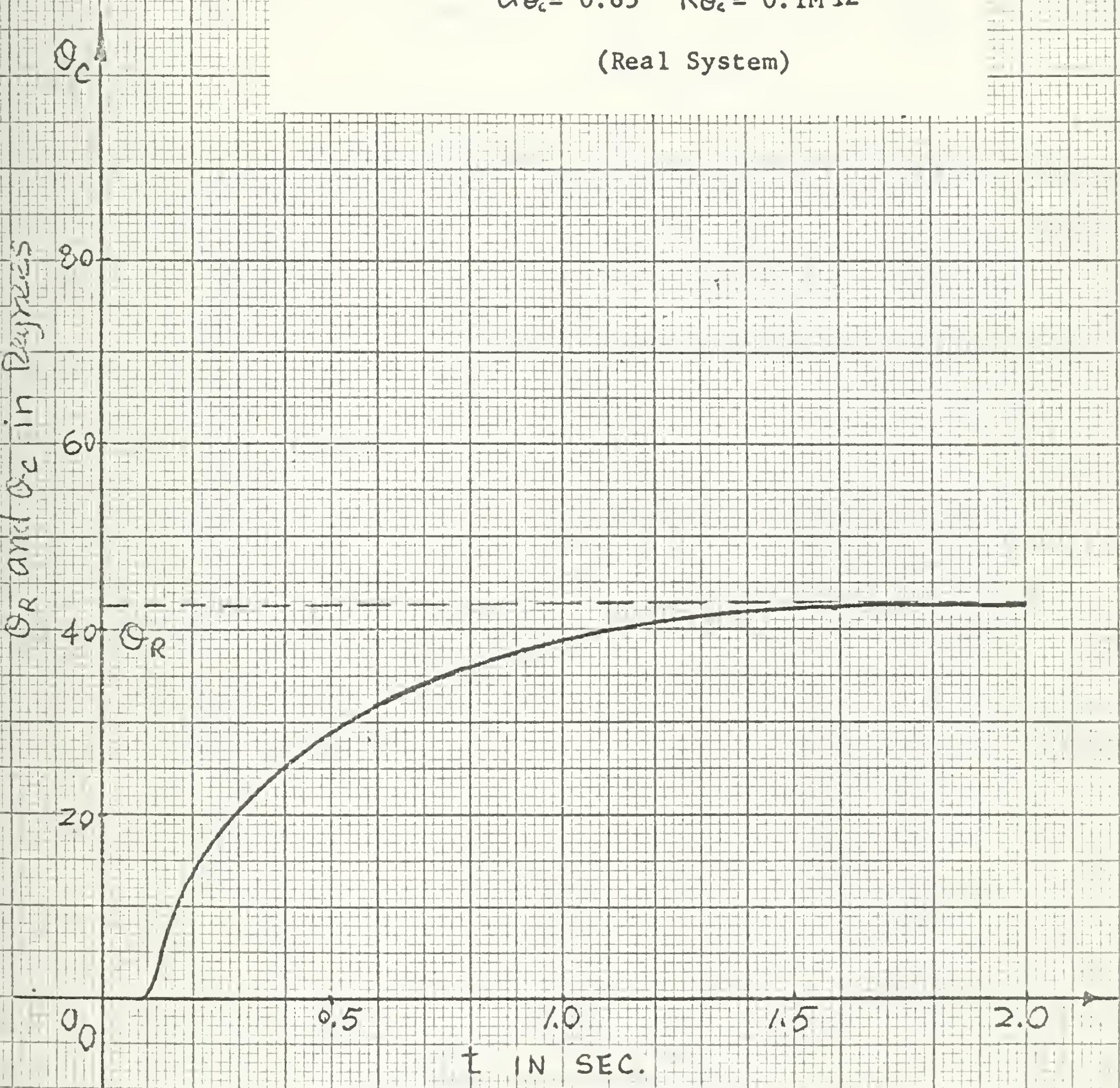


Fig. 2-57 Slow Response due to too Large $\ddot{\theta}_c$ Feedback $R_{\ddot{\theta}_c} = 0.0333\ M\Omega$, $a_{\ddot{\theta}_c} = 0.61$

(Real System)

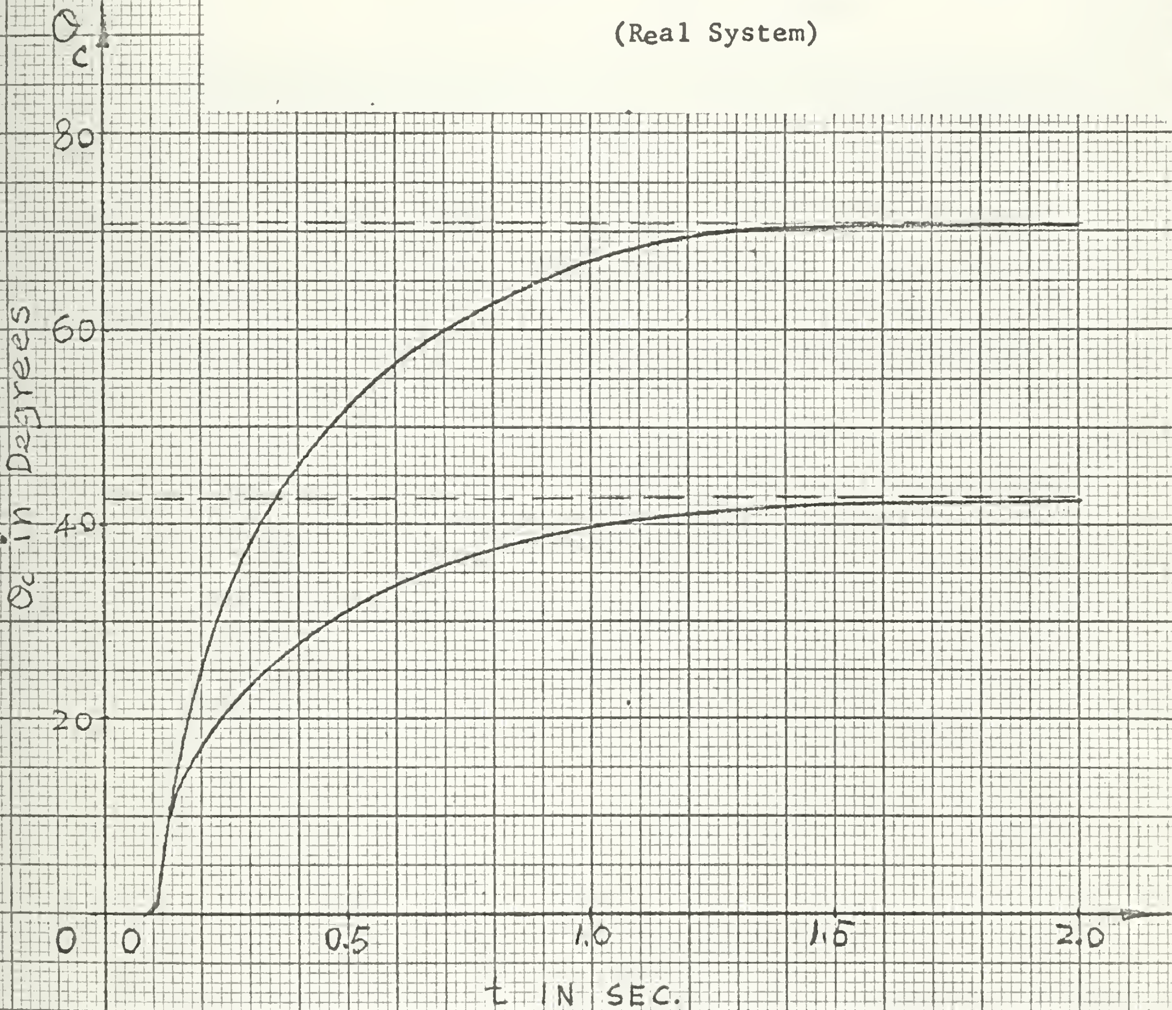
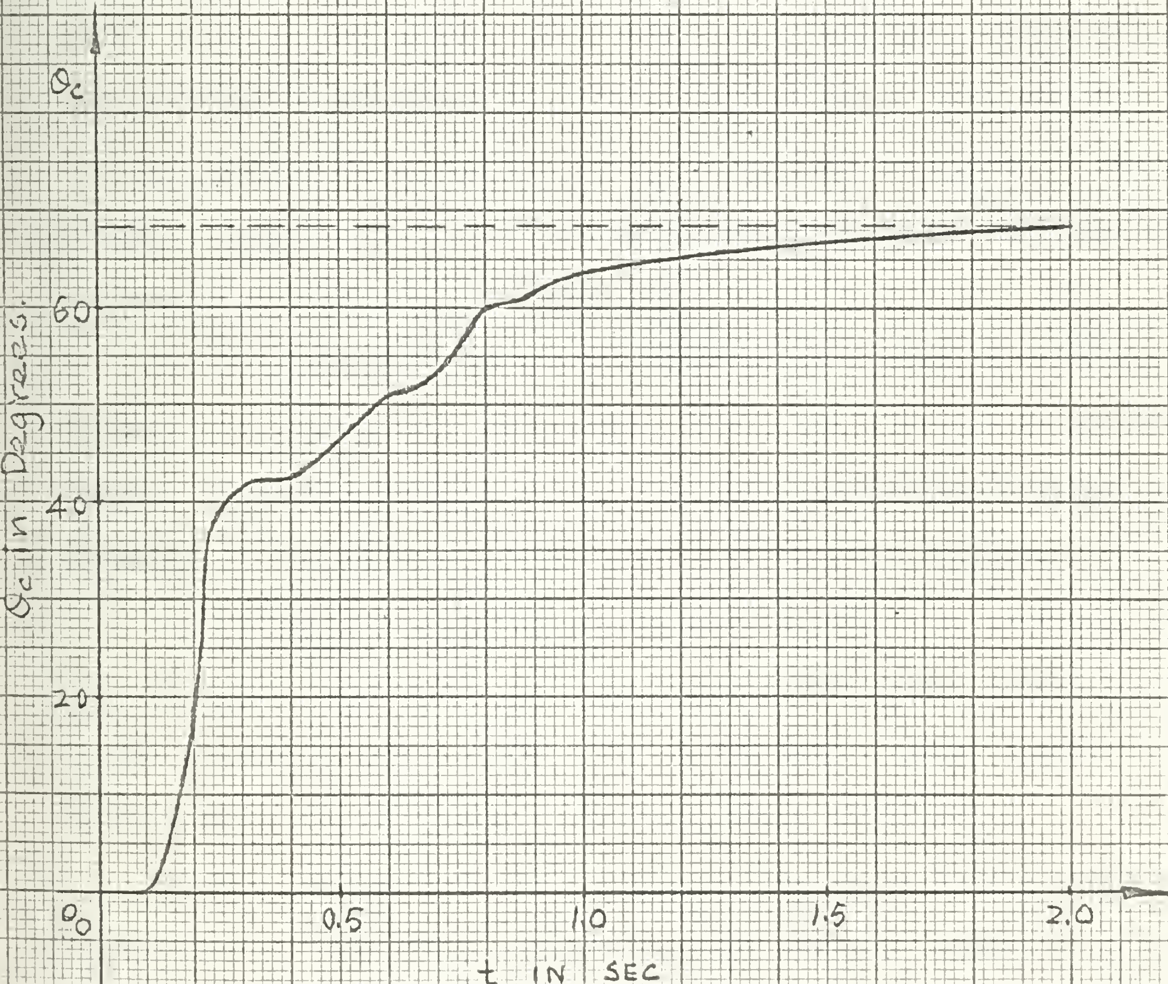


Fig. 2-58 Oscillation due to too Large $\ddot{\theta}_c$ Feedback

$a_{\ddot{\theta}_c} = 0.61$, $R_{\ddot{\theta}_c} = 0.025M$



(I) Discussions

(1) In order to use x and $\dot{x}$ to produce the control signal in this 4th order system the roots selected must make the ratio between x and $\ddot{x}$ as large as possible ($x/\ddot{x}$ large), i.e. to make the projection of the switching points in the Hyperplane very nearly in the intersection line between x-$\dot{x}$ plane and the Hyperplane. If the $\ddot{x}$ signal from the R-C differentiator does not have too much noise, then using x, $\dot{x}$ and $\ddot{x}$ to produce the U signal is better.

(2) Once a real system is set up, inherent limitations prevent a fast response, and these limitations are mostly due to nonlinearity and saturation. Since in the instant after switching a very large deceleration occurs, the amplifier or amplidyne is very easy to saturate. With the analog computer the nonlinearity or saturation can be tested from the output of the U signal, if the U signal is zero after switching then there is no nonlinearity, and the system will stay in the Hyperplane.

(3) The phase shift of the R-C differentiator that is being used to produce x, $\dot{x}$ even $\ddot{x}$, must be carefully evaluated. The way to do this is to compare the wave form for the same sine wave input to the original analog computer, or the input wave form applied to this R-C differentiator.

(4) A high gain amplifier in the main channel is recommended, because the attenuation of the R-C differentiator is very large, especially when the system response speed is fast, i.e., in the fast response system the R-C value must be small in order to keep the phase shift small.

(5) The first part of the trajectory (the system before switching) can be operated at zero damping even a slightly unstable condition, and the switching time for each system must be readjusted in order to get the best results, because in the physical system some values cannot be checked with mathematical solution perfectly.

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CHAPTER III - SATURATED INSTRUMENT SERVOS WITH DISCONTINUOUS DAMPING

Part-I GENERAL DESCRIPTION

SECTION I - INTRODUCTION

In the design of servomechanisms it is frequently necessary to use very high amplifier gains. At the same time there is a practical limit to available power supply voltages, as well as a limit to the permissible motor voltage. The net result is that the amplifier must be designed to saturate for relatively small input signals. In the case of instrument servos which position a shaft, amplifier saturation for angular errors of 1/4 degree is not an unusual situation. In addition, most instrument servo motors have a finite top speed, which may well be reached for large disturbances when the amplifier saturates easily. Thus two types of saturation frequently occur in the same instrument servo.

Unless proper damping is supplied, the saturated system is usually quite oscillatory. Compensation techniques usually rely on linear theory, and the compensated system is often more oscillatory than is desired. This chapter studies the problem of tachometer feedback compensation on the phase plane. It is proven that a fast, deadbeat response is always obtainable using continuous tachometer feedback with forward and feedback gains adjusted to utilize the eigenvectors for a critically damped or overdamped system. It is also proven that use of discontinuous tachometer feedback permits a design which approaches optimum relay servo performance.

SECTION II - EFFECTS OF SATURATION ON THE GEOMETRY OF THE PHASE PLANE

Fig. 3-1 is a block diagram showing the system to be considered in this study. The motor transfer function G(s) is restricted to be second order in this discussion, though many third order functions are permissible

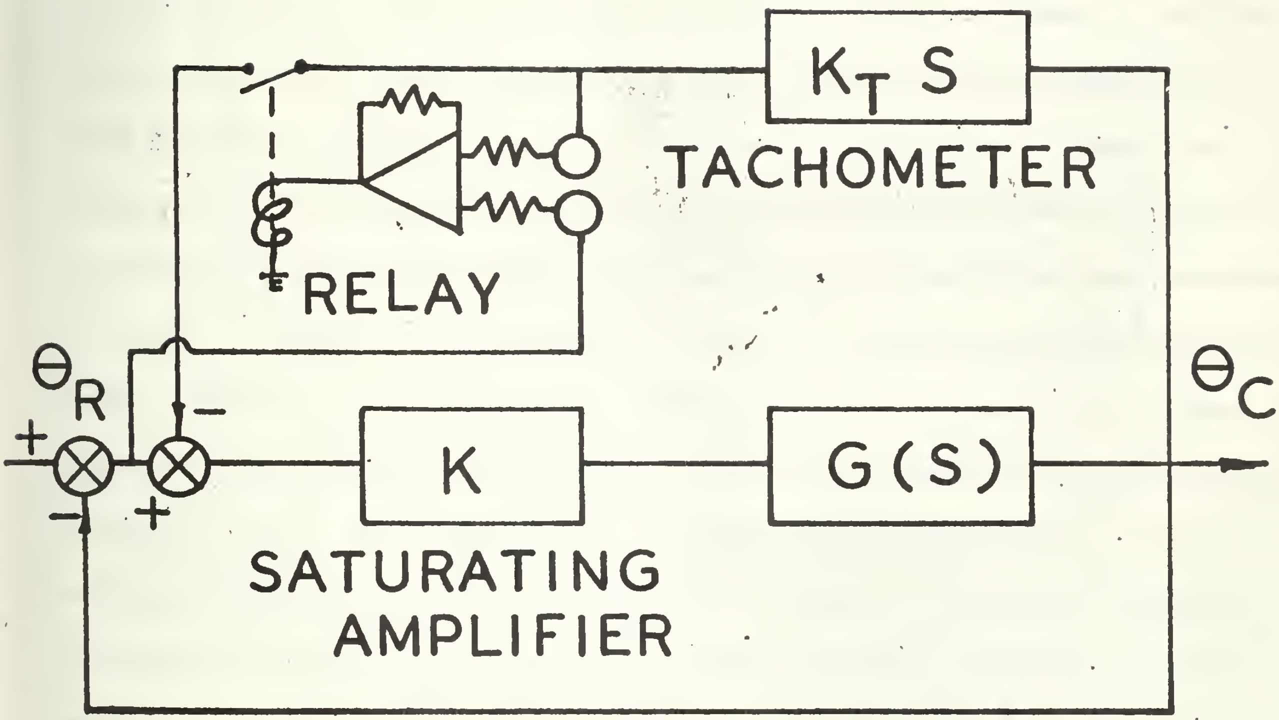

Fig.3-1 Block Diagram of an Instrument Servo with Tachometer Feedback

in practice with little effect on the results. The relay-switch arrangement in the tachometer channel permits discontinuous damping. Where continuous damping is considered in this paper the switch is assumed closed at all times.

When there is no tachometer feedback, but the main amplifier saturates for a reasonably large error the isoclines on the E vs $\dot{E}$ phase plane appear as in Fig. 3-2a. Note that the phase plane is subdivided into three areas. The center area is linear, includes the origin of the coordinate system, and has typical radial isoclines. The two outer portions of the phase plane have horizontal straight lines for isoclines, as is typical of amplifier saturation. The trajectories shown are for $G(s) = \omega^2/s^2$, which provides an undamped linear system. Note that increasing the amplifier gain decreases the width of the linear zone until, in the limit, as the gain approaches infinity the width of the linear zone approaches zero. Otherwise the phase portrait remains essentially unchanged, i.e., the singular point at the origin remains a center, the trajectory for amplifier saturation does not have a velocity limit, etc. In the more common practical case the motor transfer function is of the form $G(s) = A/s(s+a)$ and the phase portrait appears as in Fig. 3-2b. Note that in the region of amplifier saturation there is also a velocity limit, and in the linear region the singular point at the origin is a focus. The effect of raising the amplifier gain is again to decrease the width of the linear zone so that at infinite gain the width of the zone becomes zero. There is an additional effect of gain increase: in the linear zone the damping decreases as the gain increases, and as the gain approaches infinity the focus tends to a center.

From Fig. 3-2 it is readily seen that a saturated servo operated with high gain in the linear zone is necessarily poorly damped and highly oscillatory. To provide acceptable step response damping must be introduced.

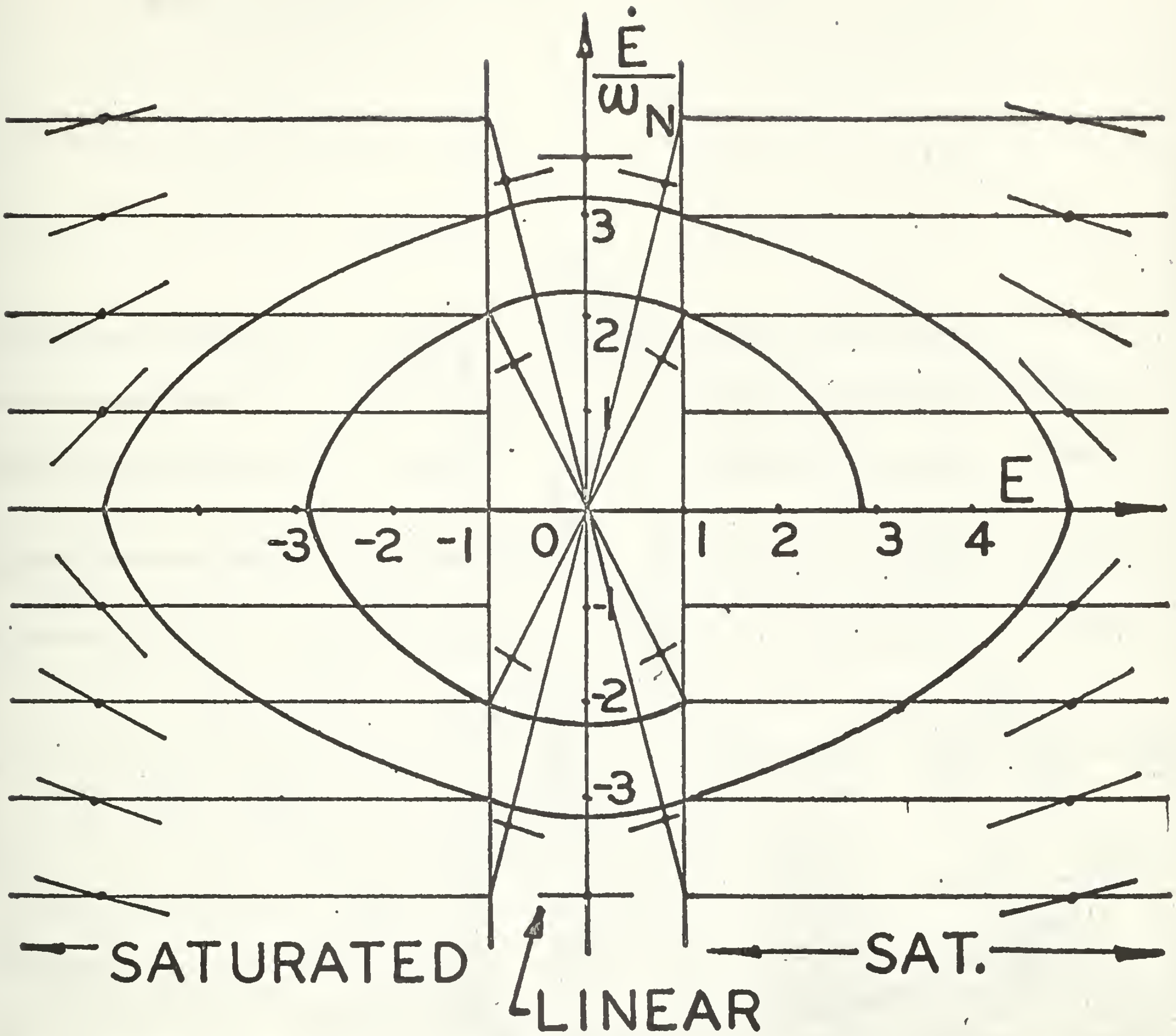

Fig.3-2a Isoclines and Trajectories for a Saturating System without Tachometer Feedback
(a) $G(s) = \omega^2/s^2$

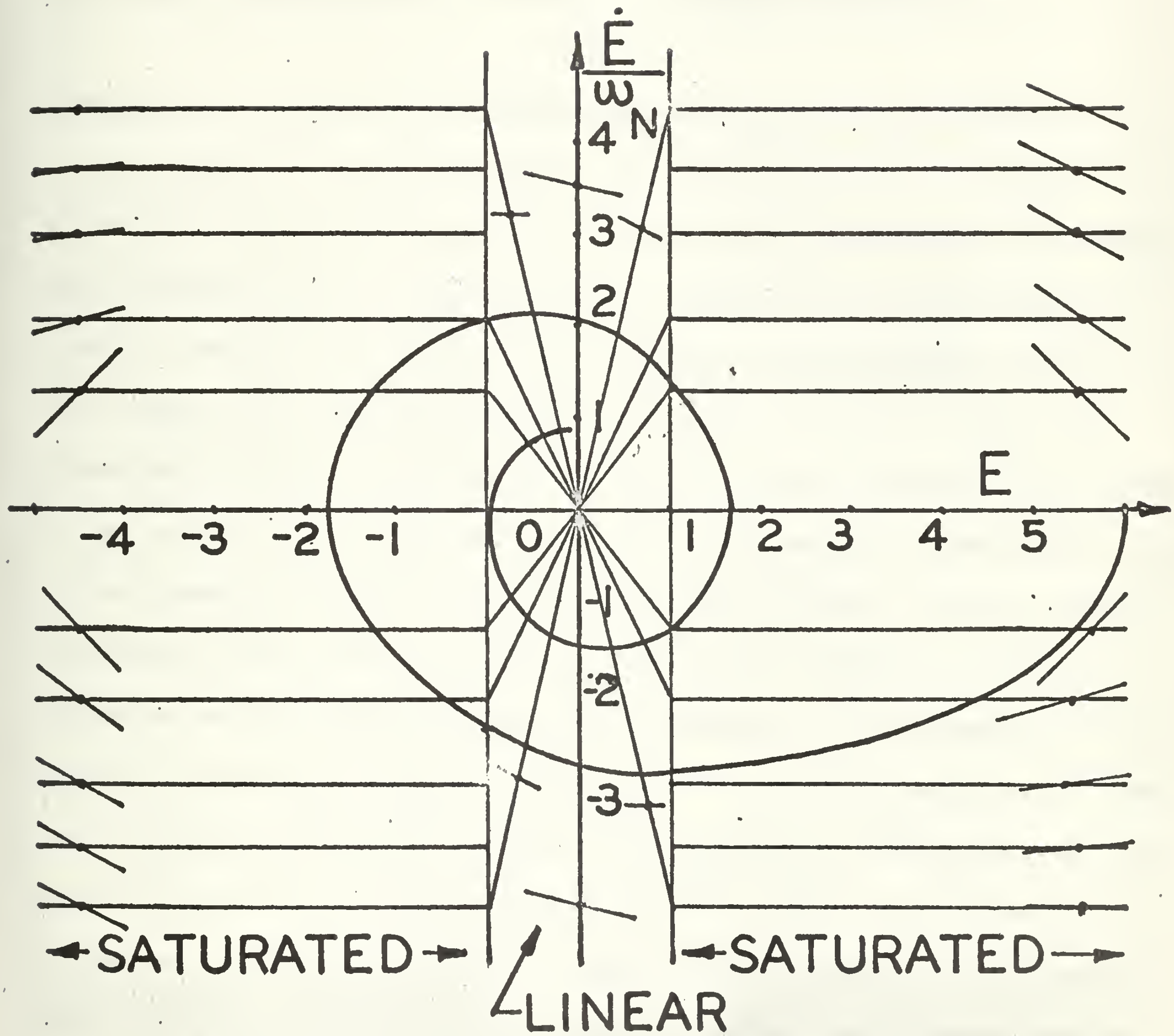

Fig.3-2b G(s) = A/s(s+a)

Mechanical dampers of the viscous and inertial types may be used, but are usually undesirable or inadequate when the gain is high. Cascaded phase lead compensators are also used, but multiple sections are required to obtain improvement in damping ratio, with the result that the final product may be appreciably underdamped. Tachometer feedback is also in common use and this study is restricted to the case of tachometer feedback.

SECTION III - EFFECTS OF TACHOMETER FEEDBACK ON THE PHASE PORTRAIT

If tachometer feedback (inverse) is introduced (switch in Fig. 3-1 permanently closed) the phase portrait is affected in two ways: the saturation dividing lines are rotated, and the damping in the linear zone is increased. This is true for both of the transfer functions used for Fig. 3-2. The rotation of the dividing lines is indicated in Fig. 3-3a, with the linear response shown underdamped. Note that the isoclines and trajectories in the saturated region are unchanged by the tachometer feedback, and the width of the linear zone (which is defined to be the horizontal distance between saturation lines because this is a measure of the magnitude of voltage required to saturate the amplifier) is not changed. As the tachometer feedback is increased the saturation lines rotate counterclockwise and the damping increases until critical damping is attained. For this condition (see Part II) the slope of the eigenvectors is $\dot{E}/E = -\omega_1$, and the slope of the saturation dividing lines is $\dot{E}/E = -1/K_t$. This condition is indicated in Fig. 3-3b for $\omega_n = 1.0$. As the tachometer feedback is increased two separate eigenvectors appear. The "fast eigenvector" rotates clockwise, while the "slow eigenvector" and the saturation lines rotate counterclockwise. The slopes of the eigenvectors, in general, are:

$$\dot{E}/E = -\omega_n(\zeta \mp \sqrt{\zeta^2-1}\ ) \qquad (3-1)$$

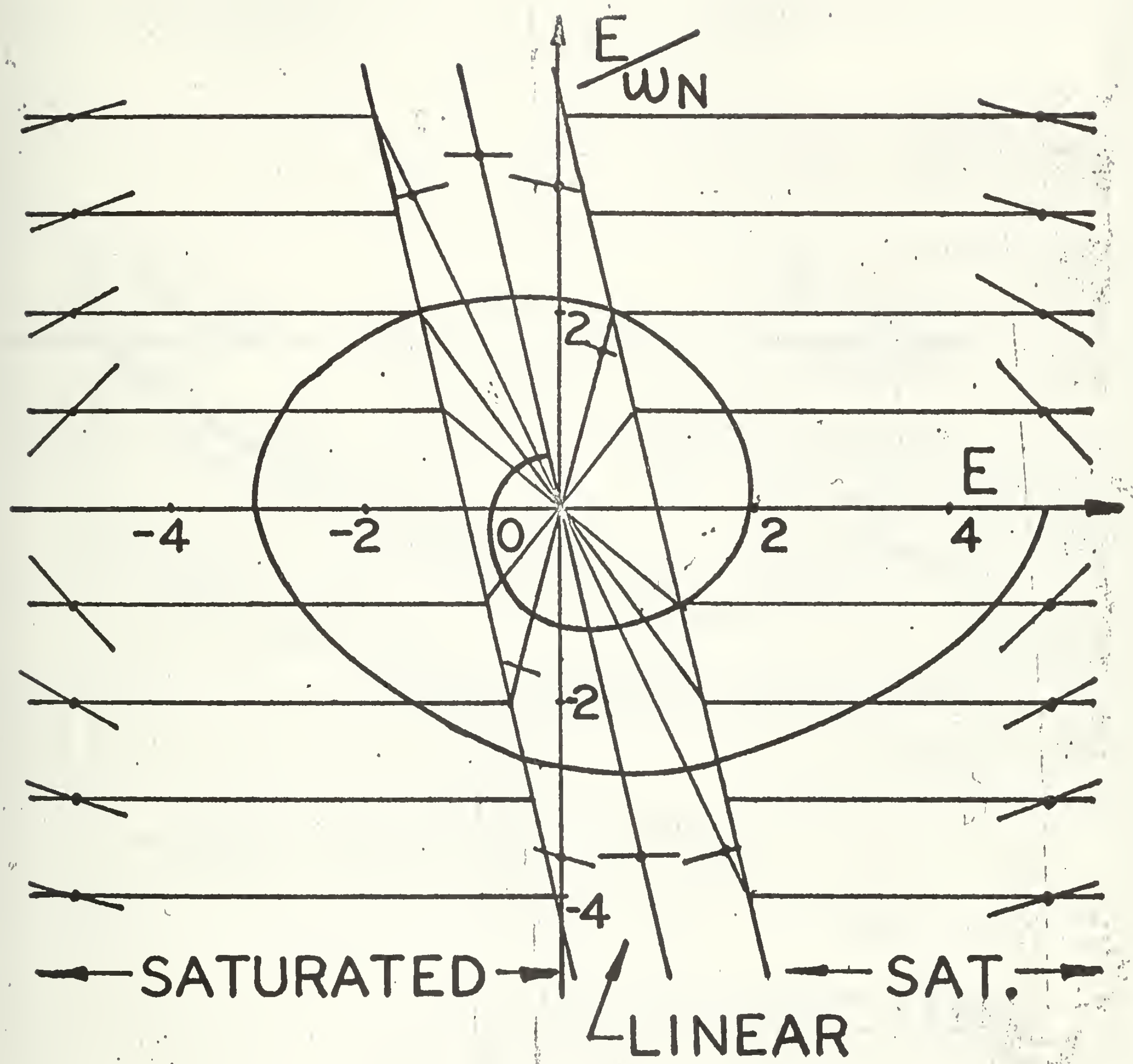

Fig.3-3a Effect of Tachometer Feedback on the Phase Portrait

(a) Saturation lines rotated and damping increased in linear zone. No change within saturated region.

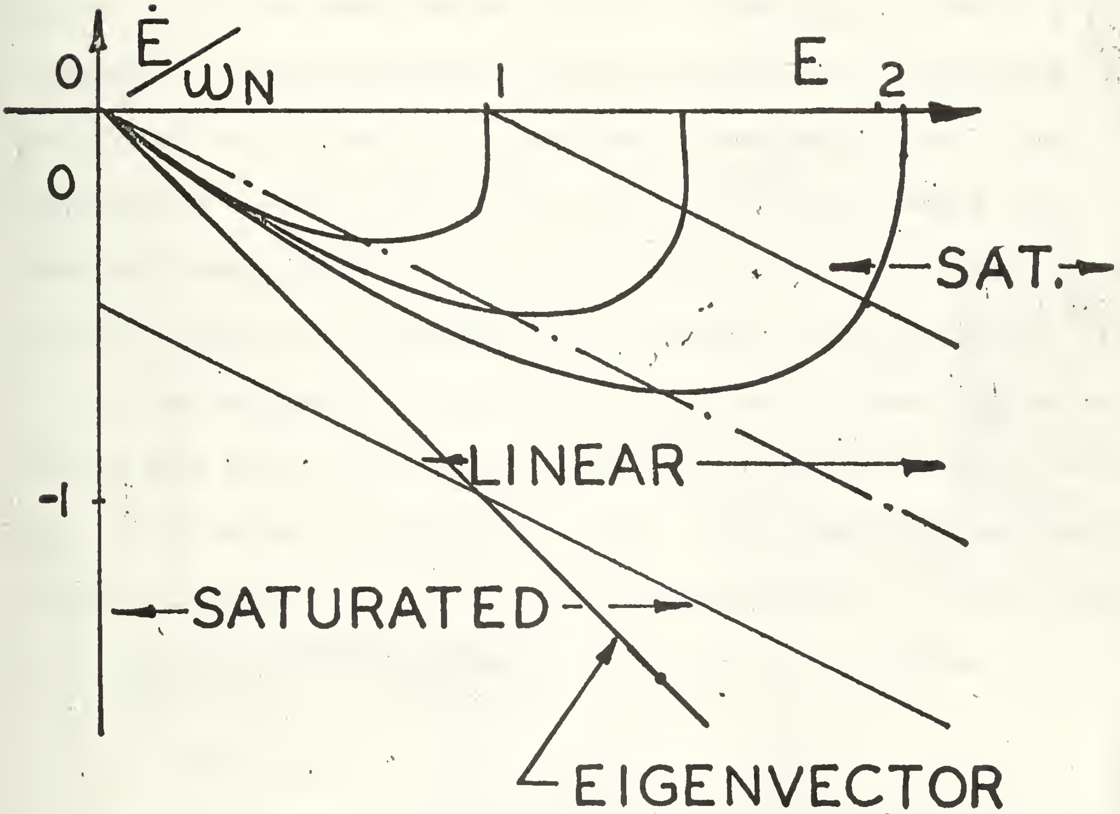

Fig.3-3b Critically Damped Case with $G(s) = 1/s^2$ and the slope of the Linear Zone Less Than That of the Eigenvector.

and the slope of the saturation lines is always

$$\dot{E}/E = -1/K_t \qquad (3\text{-}2)$$

The slope of the linear zone is adjustable, as is the slope of the eigenvectors (See Part II).

It is important to recognize the effect of the motor-load time constant. The value of this time constant controls the shape of the phase trajectory in the saturated region. Fig. 3-4 shows the influence of the time constant on the saturated phase trajectories. When the tachometer feedback is adjusted to provide critical damping or overdamping, the slope of the saturated deceleration trajectory at the limit of the linear zone is an important consideration in system adjustment, as will be shown. In general the most satisfactory response is obtained when the time constant is small.

If the tachometer feedback gain, K_t, is set at some selected value, and the main gain is increased, the width of the linear zone is decreased, ω_n is increased, and the damping ratio, ζ, may be either decreased or increased depending on the values of other parameters. If $G(s) = K/s(s+a)$ where $a = 1/\tau$, and the tachometer feedback gain is K_t, then

$$\omega_n = \sqrt{K} \quad ; \quad \zeta = \frac{a}{2\sqrt{K}} + \frac{K_t\sqrt{K}}{2} \qquad (3\text{-}3)$$

and it is readily seen that increasing K may either increase or decrease ζ depending on the initial values of a, K_t, and K. For any given a, it is always possible to obtain a specified ζ if both K and K_t are adjustable. If, for example, $\zeta = 1.0$ is desired, K is raised to narrow the linear zone, and K_t is adjusted to maintain $\zeta = 1.0$. A single eigenvector location obtains with slope increased because of the increased gain. In effect the eigenvector has been rotated clockwise, and the linear zone has been reduced

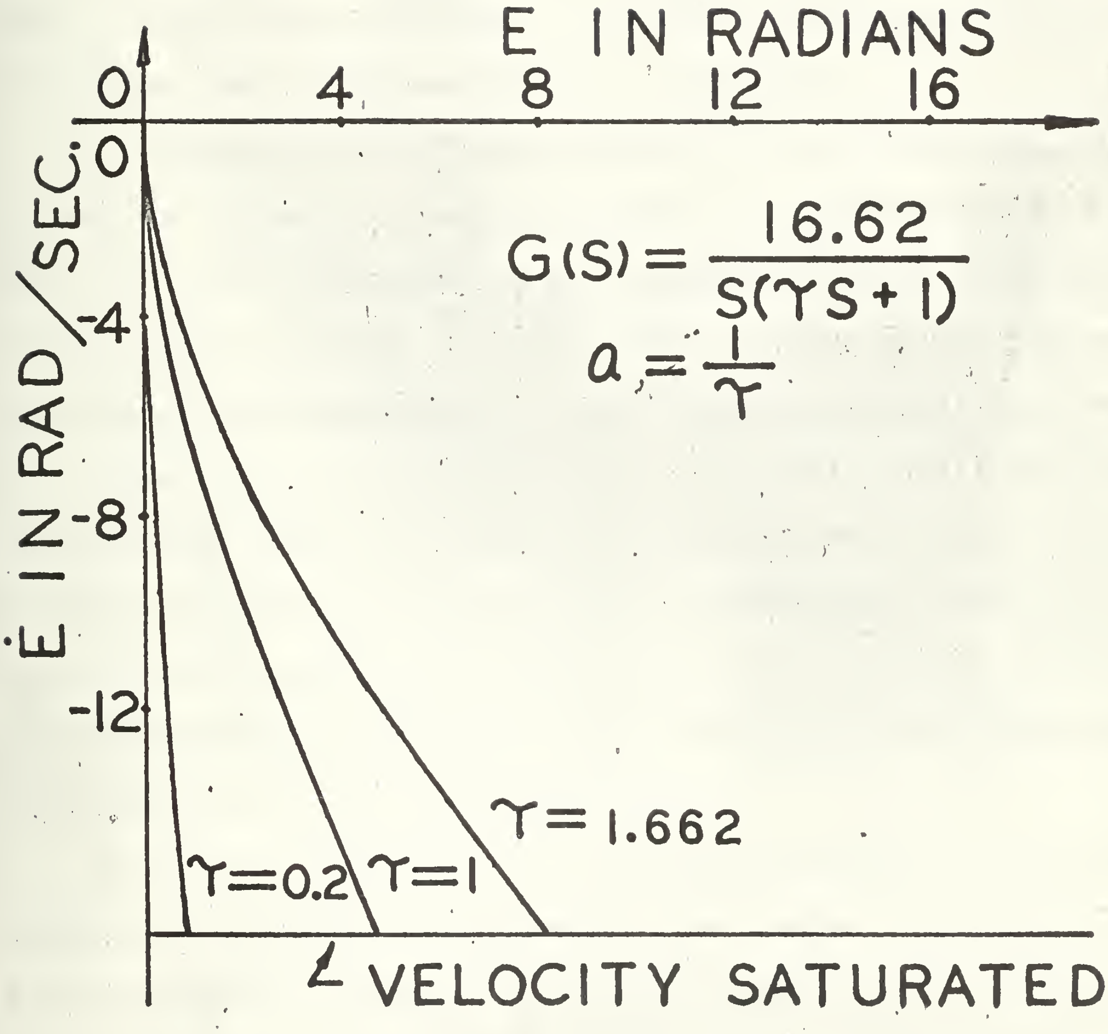

Fig.3-4 Effect of $a = 1/\tau$ on the phase Trajectory when Operation is Saturated.

in width and has also been rotated clockwise. In like manner, ζ may be increased (or decreased) if desired by simply changing the main gain K. If $\zeta > 1.0$ a fast eigenvector appears, rotated clockwise from the original position, and a slow eigenvector appears with counterclockwise rotation. The linear zone width is reduced but the linear zone is not rotated. Conversely, if the tachometer gain, K_t, is increased, ω_n is not changed, the linear zone width is not changed, but the system is overdamped and the linear zone is rotated counterclockwise. In different applications the avialability of gain may limit the choice and range of adjustments.

For a motor with a $>$ 0 there exists a velocity limit. Also, the shape of the phase trajectories when the amplifier is saturated depends on the value of a, and is independent of the tachometer feedback. Consider these facts in conjunction with the sketch of Fig. 3-3b and of Fig. 3-5a for critically damped linear operation. In Fig. 3-3b the trajectories for large initial steps are able to cross the entire linear zone, but the state point is prevented from leaving the linear zone because the direction of the saturated deceleration trajectories restricts it to the saturation boundary line. In Fig. 3-5a the eigenvector lies completely in the linear zone, and all trajectories entering the linear zone become asymptotic to the eigenvector, thus none can ever cross the linear zone.

When the motor-load pole, a, is small, the curvature of the saturated deceleration trajectory may alter conditions somewhat. Consider Fig. 3-5b. Since a is small the saturated deceleration trajectories are nearly parabolic, and for the large values of $\dot{E}$ their slope permits the state point to leave the linear zone and an overshoot results.

For the overdamped case the same concepts apply. A larger number of possible combinations exists because there are two eigenvectors. In general the following statements may be made: When a trajectory enters the linear

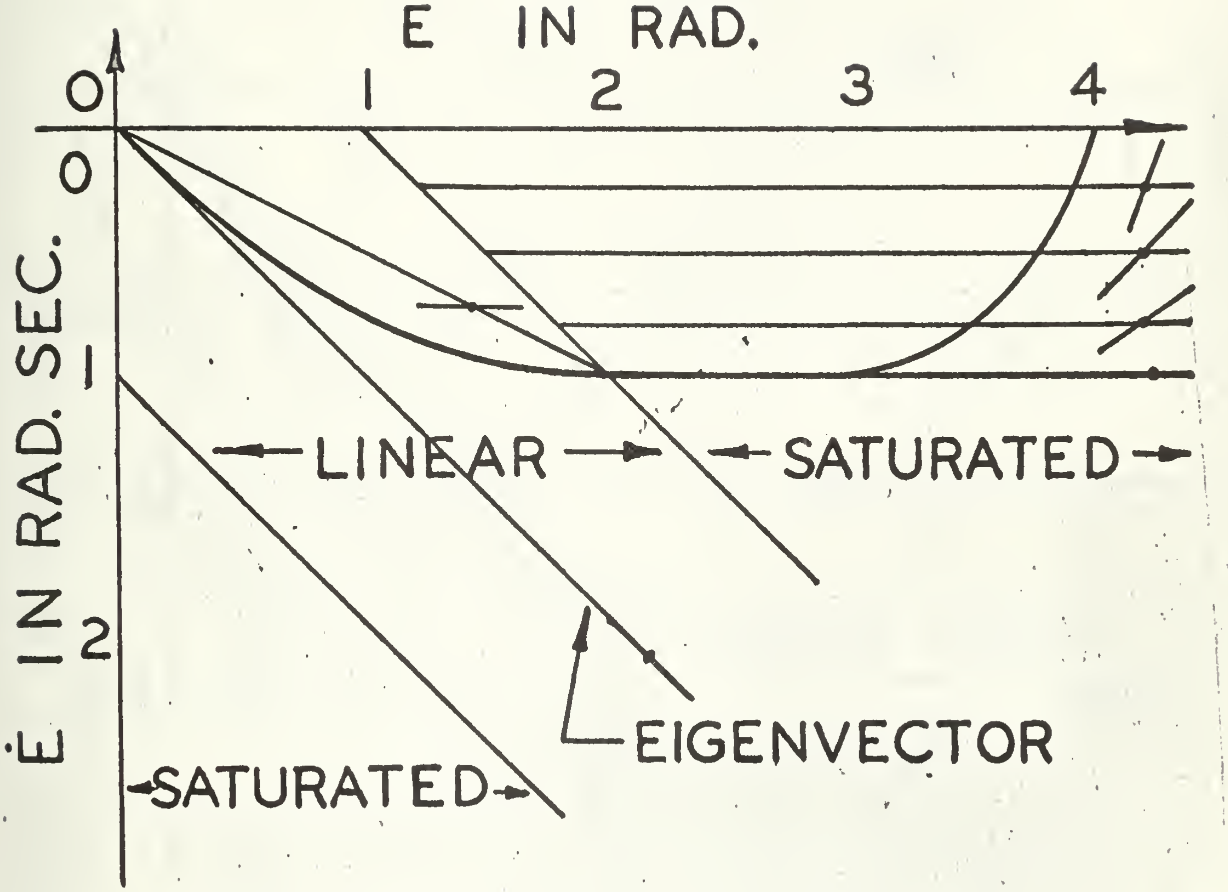

Fig.3-5a Available Performance Characteristics.

(a) Critically damped, eigenvector in the linear zone.

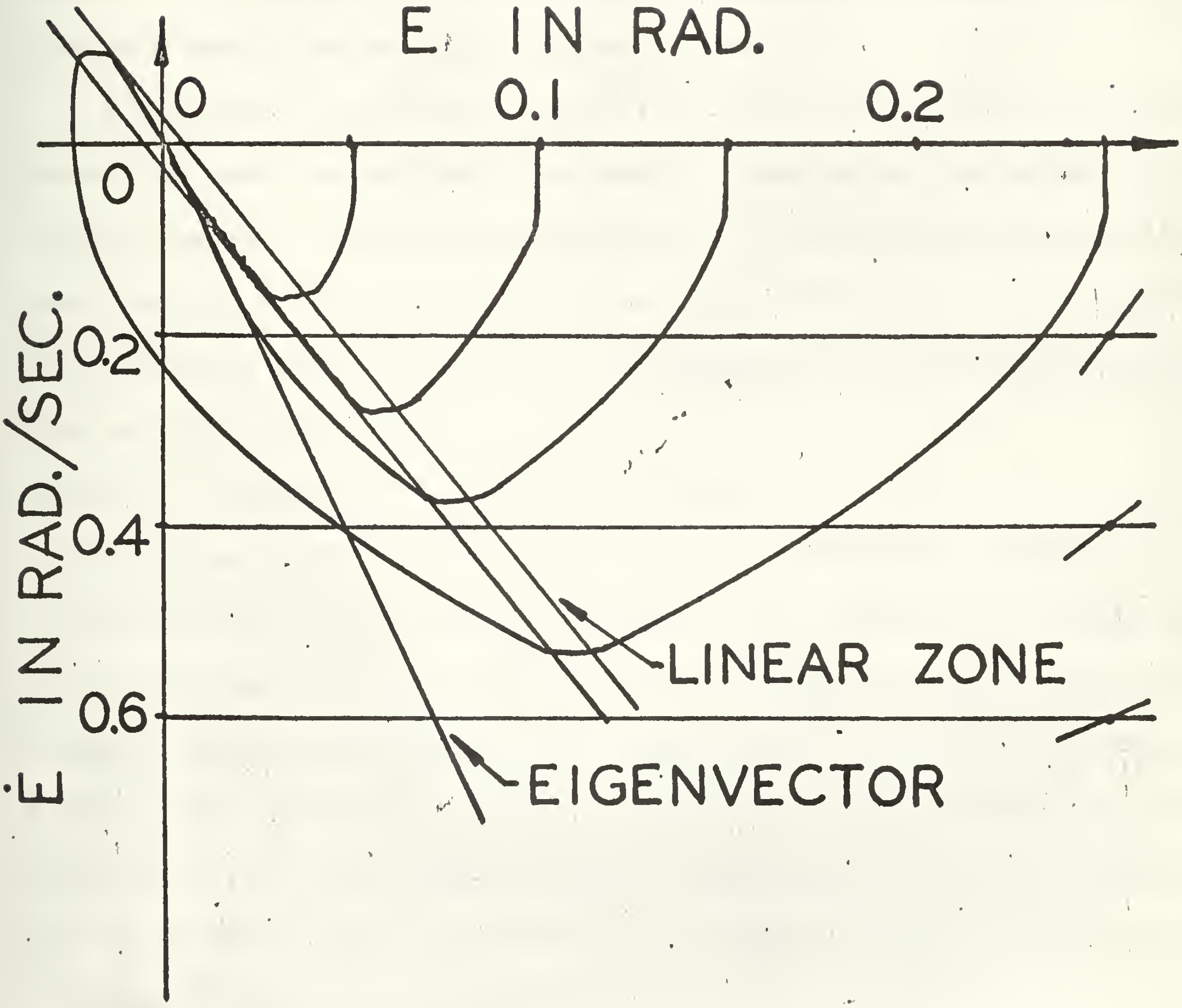

Fig.3-5b Critically damped, a = 0 (or small), deceleration trajectories permit state point to leave the linear zone.

and if an eigenvector is completely contained in the linear zone; it can not leave the linear zone if the slope of the deceleration trajectory at all points on the saturation boundary is greater than the slope of the saturation boundary line, but it can leave the linear zone if it completely crosses the linear zone and if, at its point of intersection with the saturation boundary line, the slope of the saturated deceleration trajectory is less than the slope of the saturation boundary line.

A practical illustration is given in Fig. 3-6. The transfer function numbers represent values taken from practical components. Adjustment is for critical damping. Note that the eigenvector is the center line of the linear zone, thus all trajectories approach the origin asymptotic to the eigenvector. The response is obviously very fast, and deadbeat, yet the amplifier saturates for 0.01 radian error.

SECTION IV - THE USE OF DISCONTINUOUS DAMPING

Under some conditions the use of continuous tachometer feedback may be objectionable, or the range of permissible gain values may not allow satisfactory adjustment, or the response time may not be acceptable. Many of the desirable features of an optimum relay servo may be obtained by insertion of a relay in the tachometer channel as shown in Fig. 3-1. Note that the relay is switched by a linear computer using $E \mp A\,\dot{\theta}_c$ control, so the switching line on the phase plane is a straight line extending radially from the origin. The slope of this line is adjustable and is in no way related to either the slope of the linear zone or of any eigenvector.

The operation of the system may be explained from Fig. 3-7. The switch is initially open and the linear zone is vertical, containing the $\dot{E}$-axis as shown. The width of the linear zone is established by the gain setting which in turn is determined by some specification. For a large step input the phase

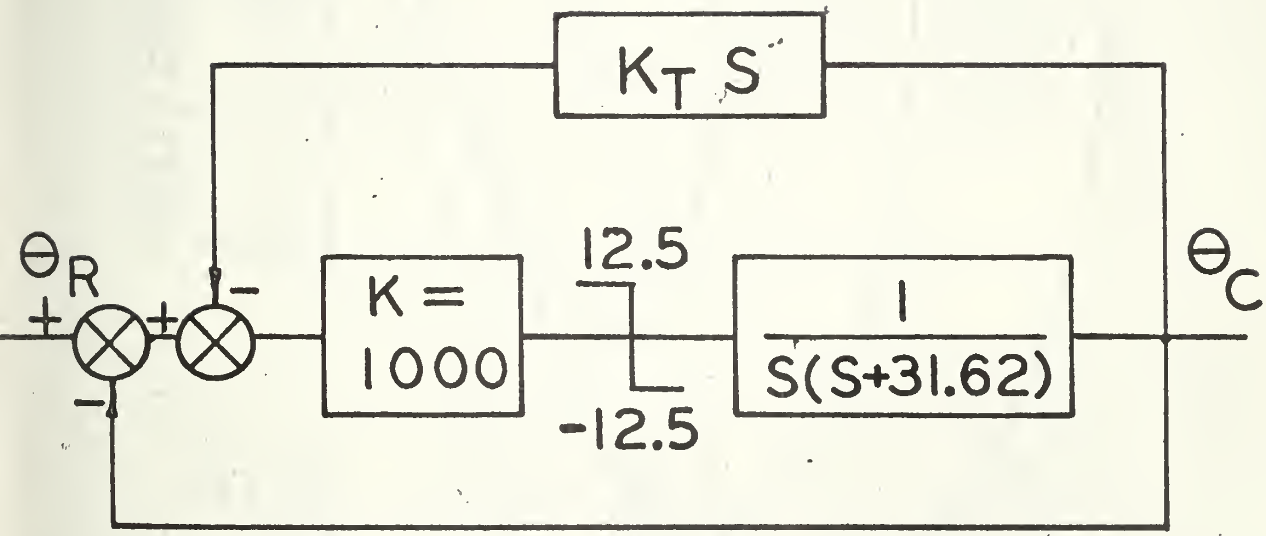

Fig.3-6a A Numerical Illustration
(a) Block diagram

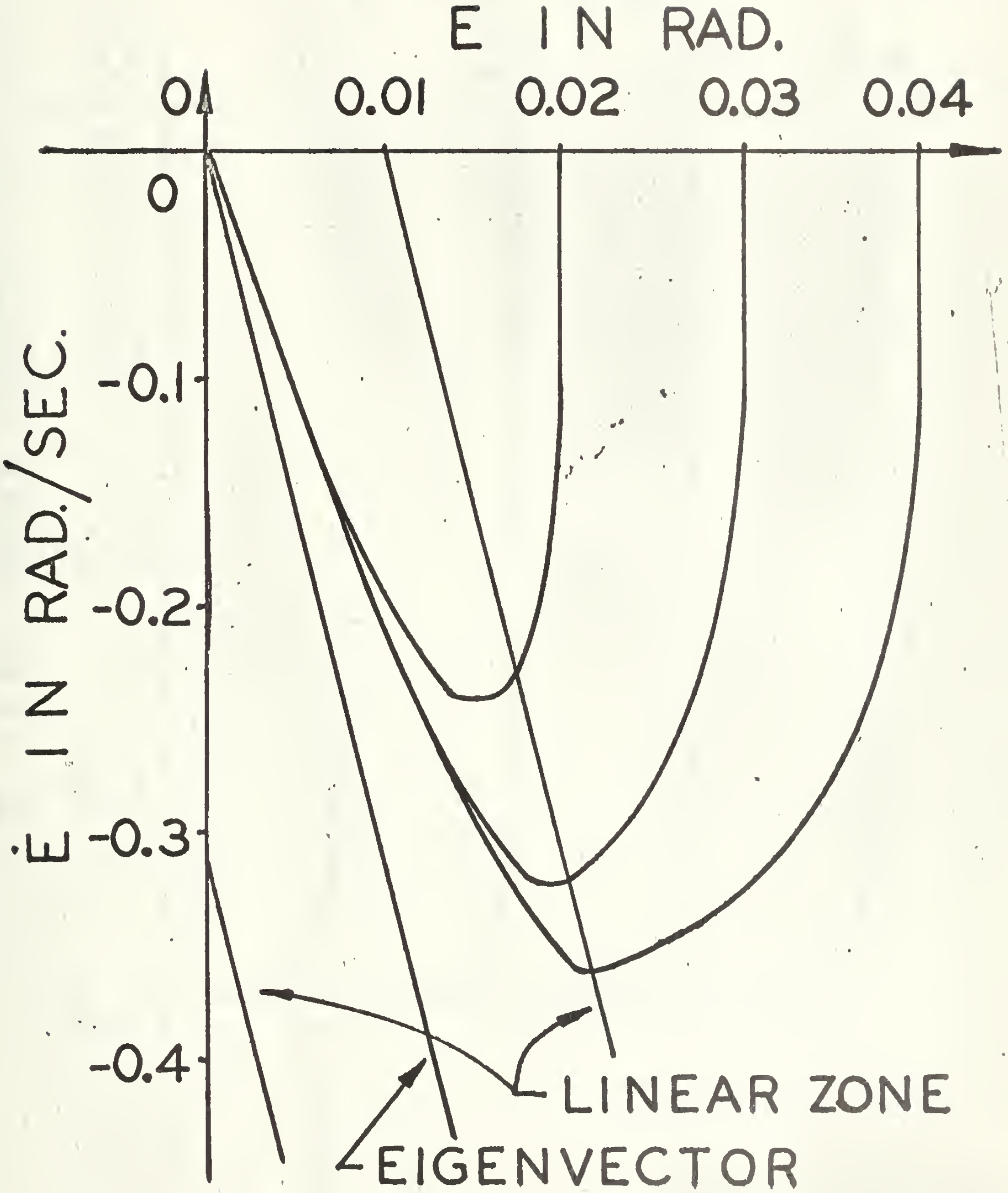

Fig.3-6b Small signal response

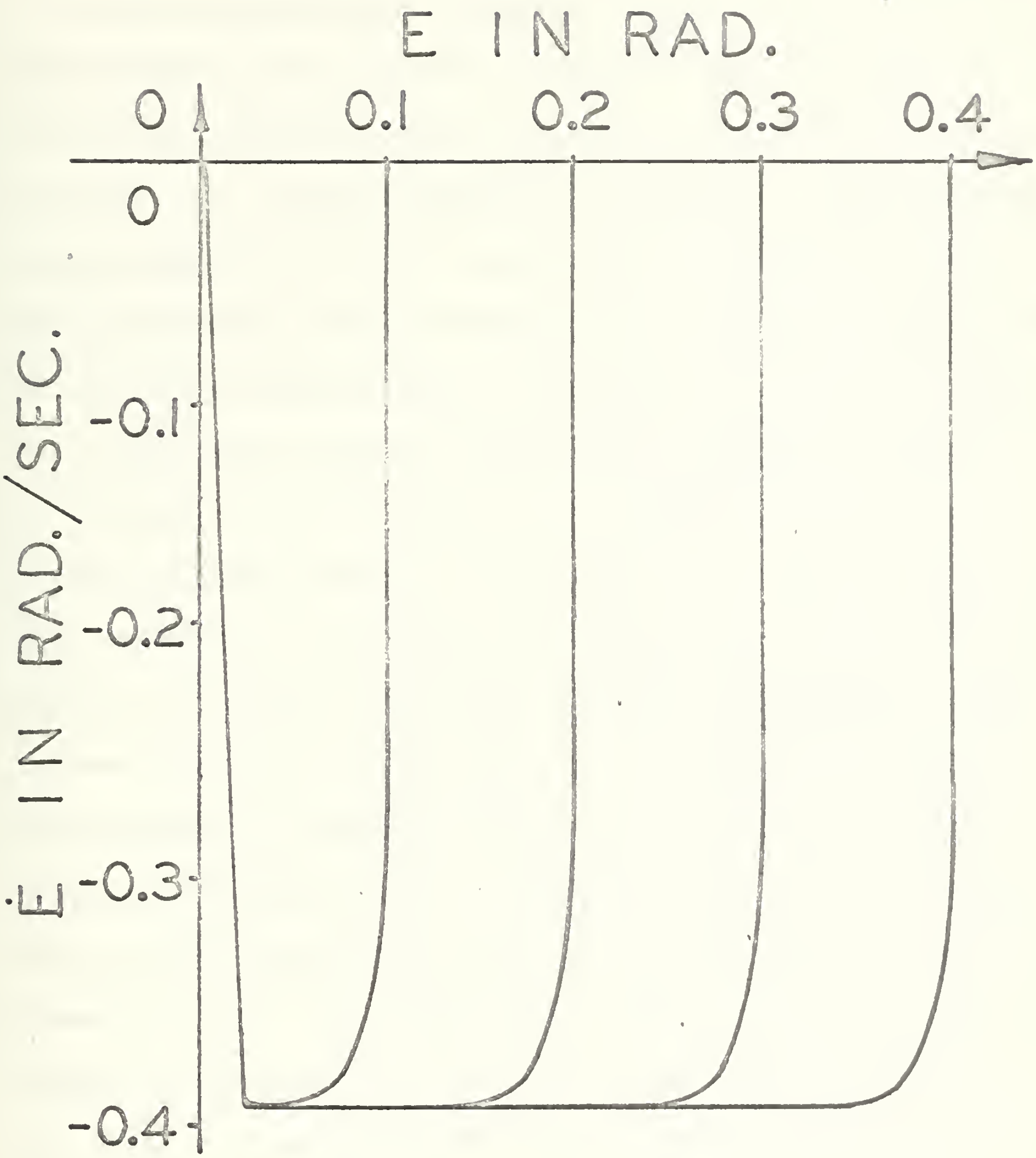

Fig. 3-6c Large singal response

trajectory might start at W, reach velocity saturation at X, and proceed at constant $\dot{E}$ from X toward Y. Assume that the tachometer gain has been set to some value such that the linear zone is rotated to a second position (as shown in Fig. 3-7) when the switch is closed. Then the trajectory from X can pass this linear zone location before the switch is operated, and if the switching line is adjusted to close the switch at some point such as X_1, the system immediately becomes saturated in reverse, and the state point transfers to a saturated deceleration trajectory. This reversal of the saturated amplifier is equivalent to the use of an ideal relay in the main transmission path, but requires only an ordinary relay or perhaps a gating circuit in the feedback path. If the switching circuit is adjusted so that the relay cannot reoperate, the trajectory enters the linear zone at X_2 and is confined to the linear zone until it reaches the origin. If the switch is adjusted to operate every time the state point corsses the switching line, then it chatters continuously as the state point follows the switching line from X_1 to the point at which the switching line enters the linear zone. If optimum relay servo performance is to be approached, a saturated deceleration trajectory is extended back from the origin until it intersects the velocity saturation line at Y. The switching line is set to pass through Y. Then the phase trajectory follows the path WXYZ, entering the linear zone at Z near the origin. The operation is clearly dual mode, a linear mode always existing near the origin. This is a desirable feature. The value of ζ for the linear zone may be chosen as desired, the only restriction being that the resulting value of K_t must locate the linear zone counterclockwise from the switching line when the switch is closed.

The performance of the system over a range of step amplitudes is of interest. For smaller steps, the switch still operates at the switching line,

but this selects a different deceleration trajectory. The maximum spread of the trajectories is readily constructed in Fig. 3-7 by establishing a second deceleration trajectory which is just tangent to the switching line, as shown. All of the deceleration trajectories for all magnitudes of step must lie within the bundle defined by the two limiting trajectories. This bundle defines the range of performance variation when the relay remains closed after switching.

From Fig. 3-7 it is seen that the action of the system is exactly that of the optimum relay servo until the trajectory bundle enters the linear zone near the origin. It is readily seen that for adjustment as in Fig. 3-7 there is never any overshoot unless it occurs in the linear zone; the maximum possible error at the point of entering the linear zone is clearly defined, and the rise time to within this limit is exactly that of the optimum relay servo.

Note that for the switching adjustment of Fig. 3-7, and a critically damped or overdamped linear region, the trajectory never overshoots, but approaches zero error asymptotically in the time domain. The maximum error upon entering the linear zone is the full width of the bundle of deceleration trajectories. If this is objectionable, but some overshoot is permissible, this maximum deviation may be halved by slightly increasing the slope of the switching line so that the trajectory bundle is translated until the origin lies at its center. Alternately, the switching circuit may be designed so that the state point follows the switch line from any point at which the deceleration trajectory intersects the switch line. This assures a much narrower bundle at the linear zone.

The width of the trajectory bundle may become excessive if the motor pole, a, is small and linear switching is used. This is illustrated in Fig. 3-8a. On the other hand, when a is large the deceleration trajectory

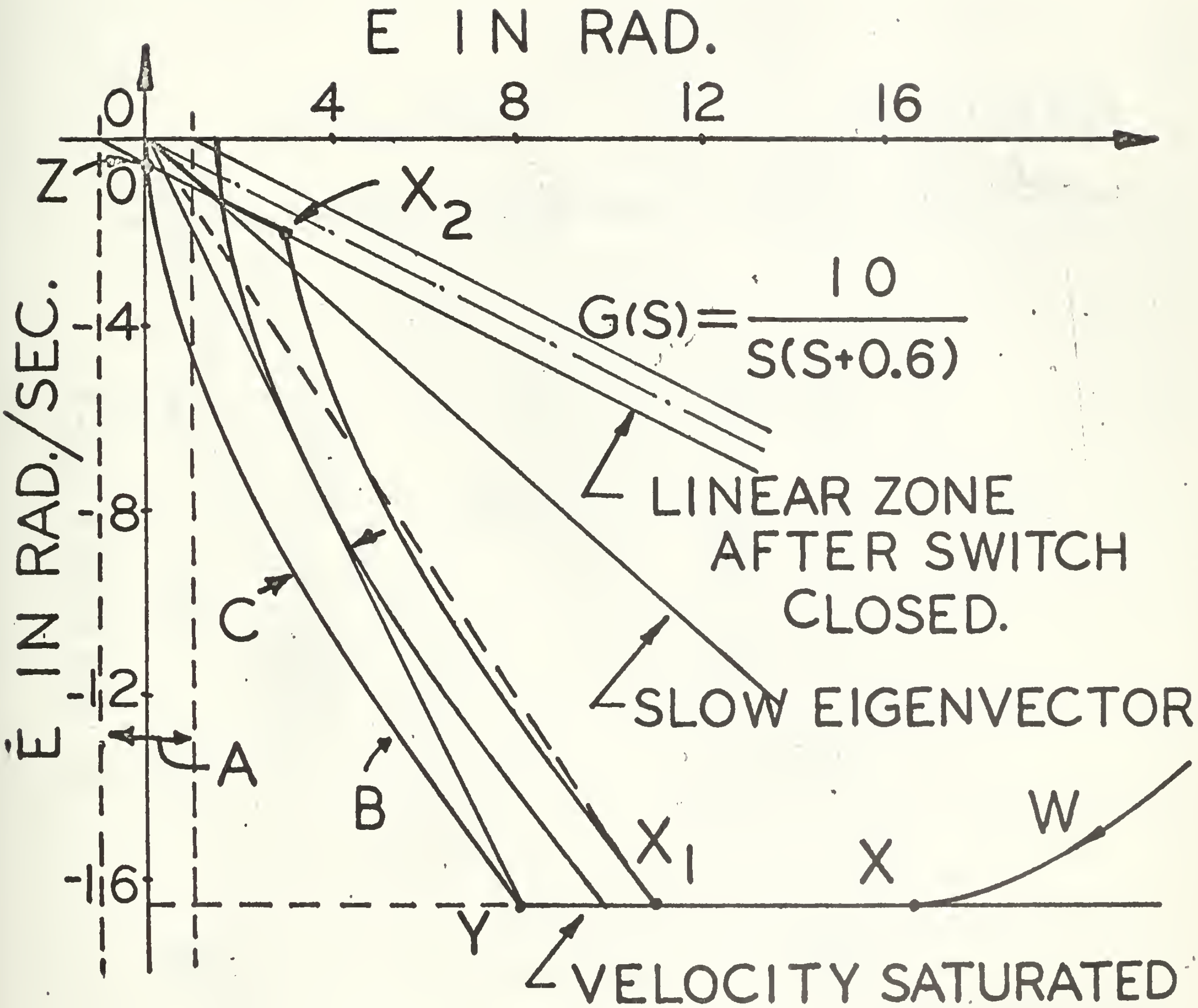

Fig.3-7. Phase Plane Analysis of Discontinuous Damping

A: location of linear zone with switch open
B: saturated deceleration trajectory
C: trajectory bundle

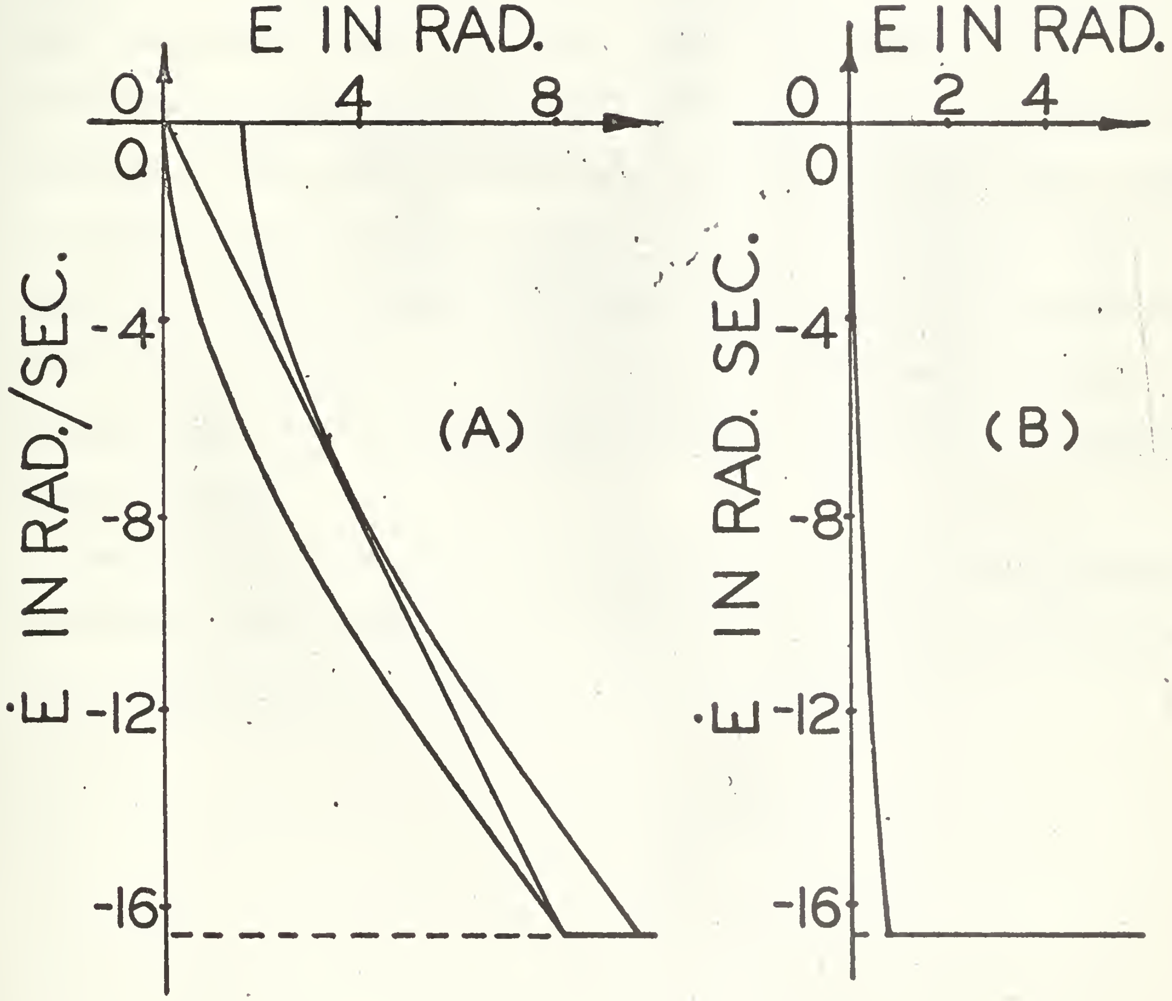

Fig.3-8. Effect of a on bundle width when $G(s) = 10/s(\tau s+1)$,
$a = 1/\tau$
a. small value of a (τ = 1.662)
b. large value of a (τ = 0.2)

is nearly a straight line and the bundle width may be very small as shown in Fig. 3-8b. When a is small and the bundle width obtained using linear switching is considered excessive, it may be reduced to any desired width by using a switching law governed by a number of straight line segments (or by permitting relay chatter). This, of course, means that the switching computer becomes more complex, since the straight line segments essentially approximate the deceleration trajectory. Still, the problem is simpler than that of building an optimum relay servo, since a finite bundle width leading into the linear zone is desired, and the number of straight line segments required and their slopes may be defined precisely. This is illustrated in Fig. 3-9. The saturation velocity and the deceleration trajectory are determined, the permissible bundle width is laid off and a second trajectory drawn. Straight line segments are then laid off within the bundle as indicated. Only two segments are needed for the bundle width of Fig. 3-9, and by inspection the width could be reduced to a smaller value if the two straight lines are properly chosen. For different specifications a larger number of segments might be required.

Fig. 3-10 gives an illustration of the results obtained with discontinuous damping. The analog computer circuit is shown in Fig. 3-10a, and the phase trajectories are shown in Fig. 3-10b and Fig. 3-10c.

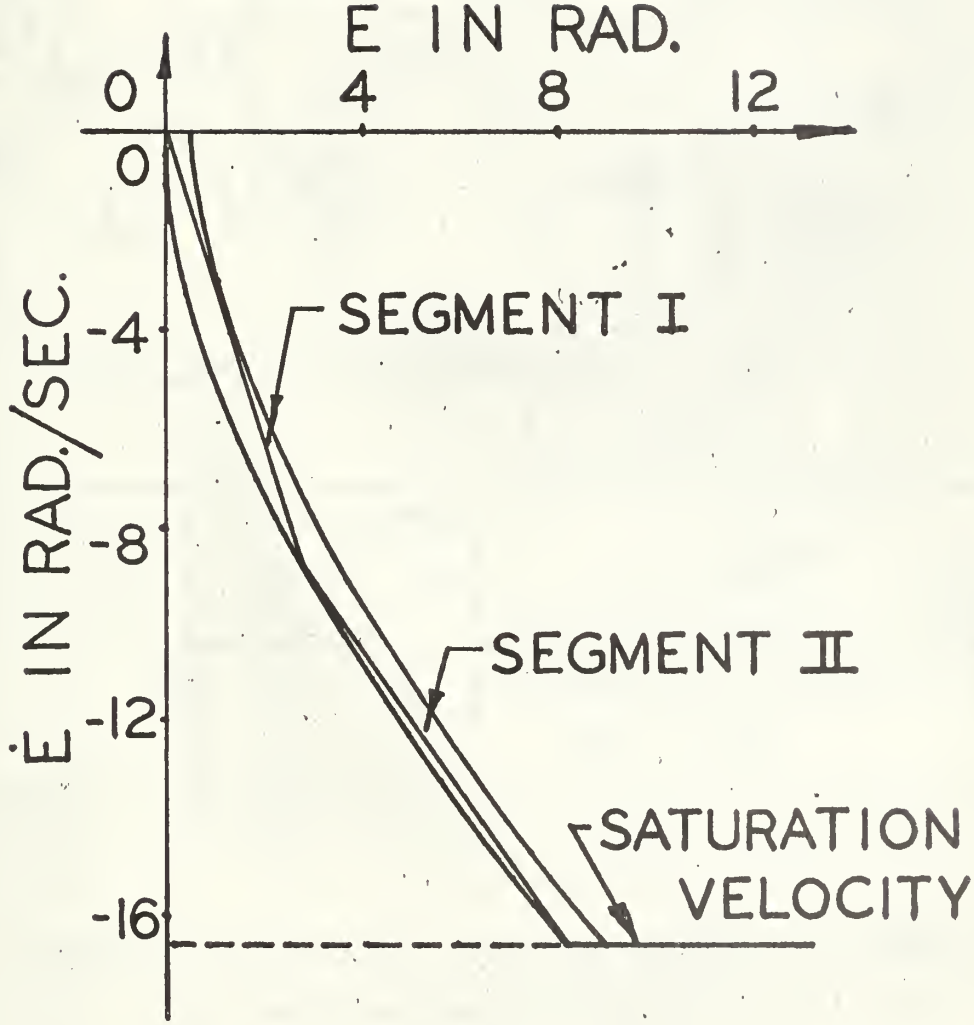

Fig.3-9 Determination of Switching Characteristics for Specified Bundle Width

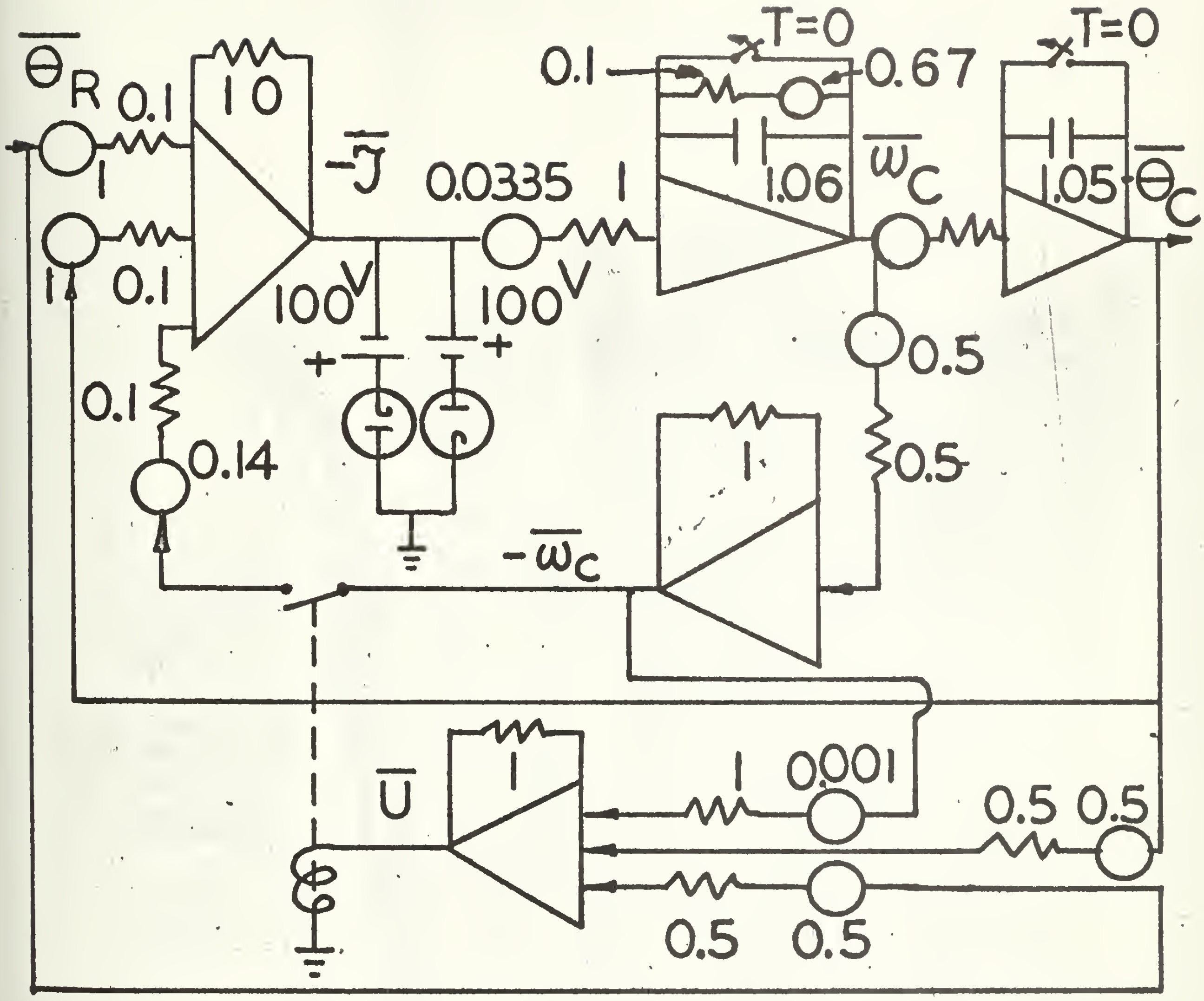

Fig.3-10a Analog Computer study of discontinuously damped system.

a. Analog computer schematic.

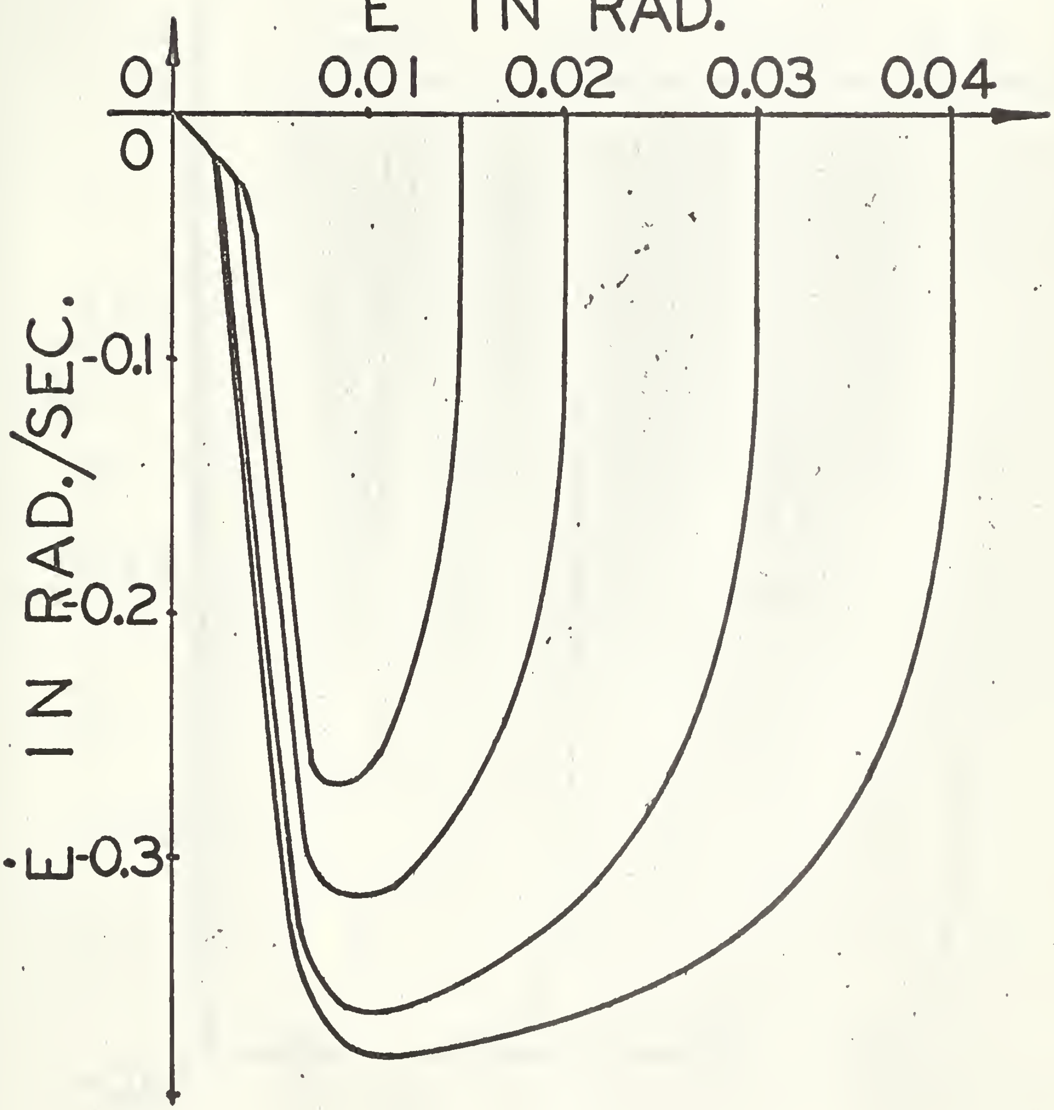

Fig.3-10b Phase trajectories, small signal

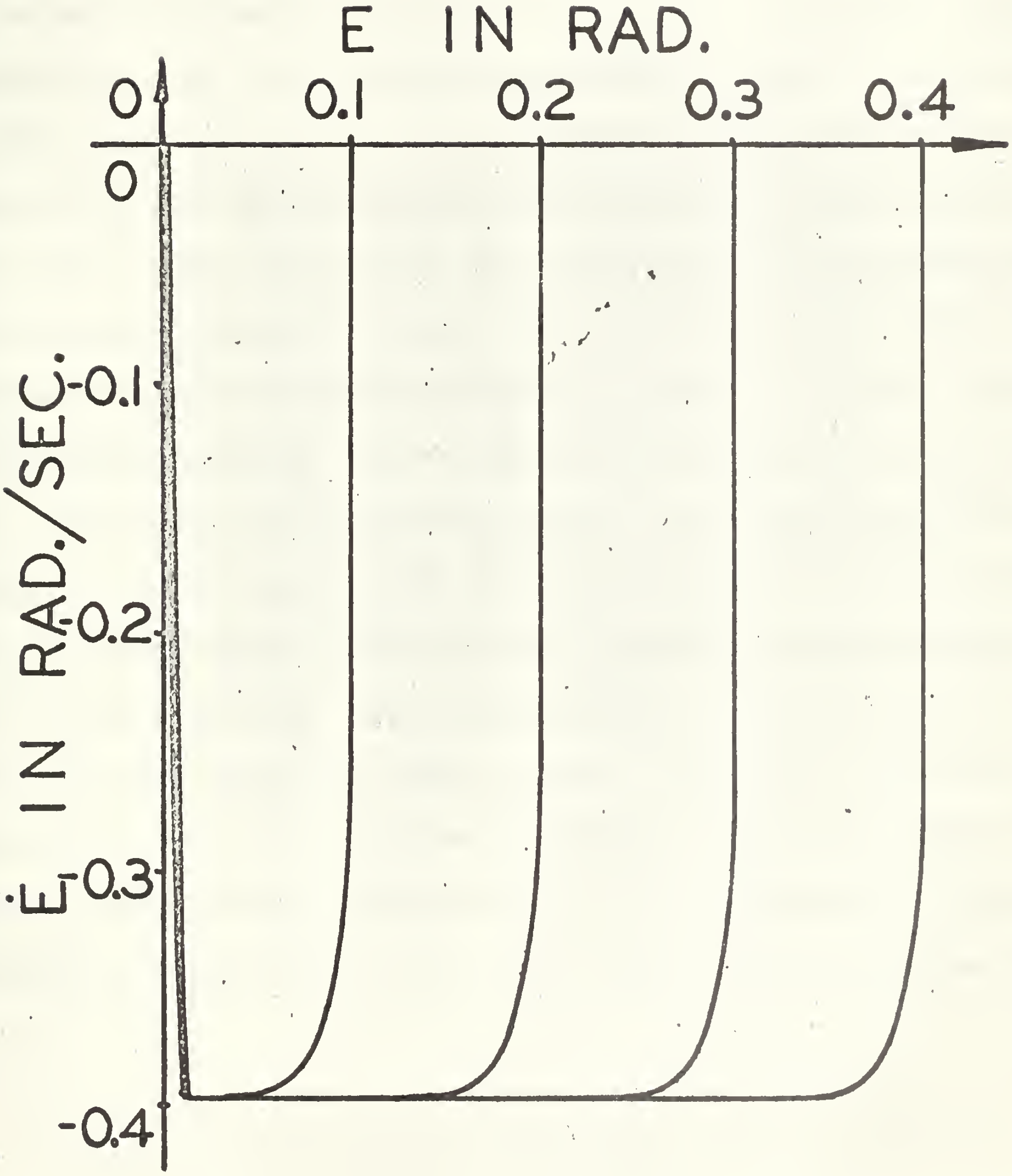

Fig.3-10c Phase trajectories, large signal.

SECTION V - COMPARISONS WITH THE OPTIMUM RELAY SERVO

The saturated servo with continuous or discontinuous tachometer feedback is a dual mode servo, and the linear zone near the origin of the phase plane suppresses limit cycles and improves static accuracy. In this sense both of the proposed schemes are superior to the optimum relay servo. The servo with continuous tachometer feedback requires no switching circuit. When discontinuous feedback is used linear switching is normally suitable and the switching computer is just an adder circuit. Thus both of these schemes are superior to the optimum relay servo in simplicity of mechanization.

The optimum relay servo is unquestionably faster, but with proper design either dual mode system approaches the response speed of the optimum relay servo. Note that in all of the methods acceleration and deceleration are obtained by varying the motor voltage. Since the optimum relay servo utilizes the maximum available voltage at all times, its speed of response is the fastest obtainable with the given motor and given saturated voltage level. The servo with discontinuous damping also utilizes maximum available voltage at all times except after entering the linear zone near the origin. If the trajectory bundle width can be made small enough the remaining error in the linear zone may be considered negligible, in which case the response speed of the discontinuously damped system is exactly the same as for the optimum relay servo. For the case of continuous tachometer feedback the response is always slower than that of the optimum relay servo because the decelerating voltage in the linear zone is never the full saturated value.

SECTION VI - DESIGN PROCEDURES, CONTINUOUS TACHOMETER FEEDBACK

The first step in all of the design procedures suggested here is to set the main gain to meet performance specifications. This determines the width of the linear zone.

Next construct the saturated deceleration trajectory and compute the maximum velocity. Measure the slope of the deceleration trajectory at the maximum velocity point and set K_t to give a linear zone with this slope. This guarantees that the state point can never cross and emerge from the linear zone. Since a, K and K_t are specified by these computations, the damping in the linear zone is readily evaluated and system performance estimated. The major consideration is the slope of the linear zone itself. For small a the slope may not be great and response may be slow.

If faster response is required or a larger ζ is needed, then both K and K_t must be altered. The linear zone must be rotated clockwise to make the system faster, so K_t must be reduced. This may permit the phase trajectory to cross the linear zone (for large steps) with consequent possibilities of overshooting. Changing K changes ζ and the width of the linear zone. For $\zeta \geqq 1.0$ the eigenvector locations are altered. If the slow eigenvector lies entirely in the linear zone for velocities up to saturation velocity no overshoot is possible.

Fastest response, approaching that of the optimum relay servo, is obtainable when the motor-load pole a is large. From equation (3-3) it is seen that for $K_t = 0$, $\zeta > 1$ if $a > 2\sqrt{K}$. Then increasing K_t rotates the fast eigenvector clockwise and the linear zone counterclockwise, so K_t can be selected to place the linear zone at any location with respect to the eigenvectors.

It is apparent that for various values of a and various gain requirements suitable response characteristics may not be available. In such cases the use of discontinuous tachometer damping may be helpful.

SECTION VII - DESIGN PROCEDURES, DISCONTINUOUS DAMPING

The main gain, deceleration trajectory, and maximum velocity must be evaluated as before. Then draw a straight line from the origin to the point of intersection between the optimum deceleration trajectory and the saturation velocity line. Next evaluate the bundle width. If the bundle width is satisfactory linear switching is used, the slope of the switching line is evaluated and the switching computer designed. If the bundle width is excessive, it may be reduced by using a switching circuit which permits the relay to chatter along the switching line, or by designing a computer using several straight line segments. Next choose K_t so that the slope of the linear zone is less than that of the switching line, but check the damping which this value produces in the linear zone to be sure that operation in the linear mode is satisfactory.

SECTION VIII - RAMP RESPONSE

When a ramp input is used and the system is designed for continuous tachometer feedback, the steady state velocity lag error tends to increase with the amount of tachometer feedback. On the phase plane the linear zone is translated so that its center lies at $E = K_t \omega_i$ and the focal (or nodal) point is translated to $K = (a + KK_t)\omega_i/K$, where ω_i is the magnitude of the ramp input. For small values of a the steady state error lies within the linear zone, and thus is somewhat greater than $K_t \omega_i$. For large values of a the nodal point lies outside the linear zone in the saturated zone and is therefore a virtual node. The state point therefore comes to rest on the boundary of the linear zone such that $E_{ss} = K_t \omega_i$ + 1/2 linear zone.

When discontinuous damping is used the switch can be opened as steady state is approached. This sets K_t to zero, and the linear zone returns to the origin. The steady state error obtainable is one half the linear zone width or less. (The further investigation will be given in Chapter V).

SECTION IX - TESTS OF AN EXPERIMENTAL SERVO

Laboratory tests were made on an instrument servo in which the drive element was a D. C. shunt motor coupled to a D. C. tachometer. Both continuous and discontinuous damping techniques were used. The results were as predicted by the theory.

SECTION X - CONCLUSIONS

The step response of second order servos with amplifier saturation has been analyzed using phase plane methods. It has been shown that the use of tachometer feedback can provide deadbeat response, and that the tachometer may be connected permanently in feedback, or may be inserted discontinuously. It has been shown that a deadbeat response is assured when tachometer feedback is continuous if the slope of the linear zone is less than the minimum slope of the saturated deceleration trajectory, but that deadbeat response with shorter rise time is available by assuring certain geometric relationships between the linear zone and the eigenvectors.

Use of discontinuous damping permits design of a dual mode servo which has very nearly optimum relay servo response. Deviation from optimum response occurs only within the linear zone of the dual mode, and this deviation can be minimized by proper design of the switching circuit.

PART II - DETAIL CALCULATIONS COMPUTER SETUP EQUATIONS AND EXPERIMENT RESULTS.

SECTION I - RELATIONSHIPS FOR THE SLOPES OF THE EIGENVECTORS AND THE SLOPE OF THE LINEAR ZONE IN THE SECOND ORDER SYSTEM WITH TACHOMETER FEEDBACK.

The characteristic equation may be written as

$$\ddot{E} + 2\zeta\omega_n\dot{E} + \omega_n^2 E = 0 \tag{3-4}$$

or, using hardware parameters as in Fig. 3-1

$$\ddot{E} + (a + K_t K)\dot{E} + KE = 0 \tag{3-5}$$

and by comparison $K = \omega_n^2$ and $a + K_t K = 2\zeta\omega_n$ from which $\omega_n = \sqrt{K}$ and $\zeta = (a + K_t K)/2\sqrt{K}$.

The isocline equation is

$$\frac{\ddot{E}}{\dot{E}} = \frac{d\dot{E}}{dE} = N = \frac{-2\zeta\omega_n\dot{E} - \omega_n^2 E}{\dot{E}} \tag{3-6}$$

from which

$$\frac{\dot{E}}{E} = \frac{-\omega_n^2}{N + 2\zeta\omega_n} \tag{3-7}$$

Since $\dot{E}/E$ is the slope of the isocline, and for the eigenvector the slope of the isocline is identical with the slope of the trajectory, then the desired condition is $\dot{E}/E = N$. Thus

$$N^2 + 2\zeta\omega_n N + \omega_n^2 = 0 \tag{3-8}$$

$$N = -\omega_n(\zeta \mp \sqrt{\zeta^2 - 1}) \tag{3-9}$$

For critical damping $\zeta = 1.0$ and thus

$$N_c = -\omega_n \tag{3-10}$$

Substituting system parameter values for the critically damped case only, it may be noted that

$$N_c = -\omega_n = -\sqrt{K} = -\left(\frac{a + K K_t}{2}\right) \qquad (3\text{-}11)$$

from which it is required that

$$-\frac{1}{K_t} = \frac{-K}{2\sqrt{K} - a} \qquad (3\text{-}12)$$

Digressing momentarily to the slope of the saturation lines, the amplifier saturates when the magnitude of its voltage is some value

$$\left| e_{sat} \right| = E - K_t \dot{\Theta}_c = E + K_t \dot{E} \qquad (3\text{-}13)$$

which is the equation of a straight line.

Putting in slope form

$$\dot{E} = -\frac{1}{K_t} E + \left| e_{sat} \right| \qquad (3\text{-}14)$$

from which the slope on the $\dot{E}$ vs E plane is

$$\frac{\dot{E}}{E} = -\frac{1}{K_t} \qquad (3\text{-}15)$$

From equations (3-11), (3-12) and (3-15) it is readily seen that the slope of the saturation lines (which define the linear zone) is seldom the same as the slope of the eigenvector for critical damping. There are three cases:

1. If a is small, or zero $-\frac{1}{K_t} \cong \frac{-\sqrt{K}}{2} = \frac{-\omega_n}{2}$ and the slope of the linear zone is one half the slope of the eigenvectors.

2. If $a = \sqrt{K}$ then from equation (3-12), $-\frac{1}{K_t} = -\sqrt{K} = -\omega_n$ and the eigenvector is parallel to the saturation lines and in the middle of the linear zone.

3. If $a > \sqrt{K}$, $\frac{-1}{K_t} = \frac{K}{2\sqrt{K} - k\sqrt{K}} = \frac{\sqrt{K}}{2 - k}$ where $k > 1$, thus $-\frac{1}{K_t} = \frac{\sqrt{K}}{2 - k} > K$ and the slope of the linear zone is greater than the slope of the eigenvectors.

Case 1 is illustrated by Fig. 3-3b. Case 2 is illustrated by Fig. 3-6.

When $\int > 1$ equation (3-9) does not reduce to a convenient form, but it is apparent that for many cases the linear zone may be located at any desired position with respect to the eigenvectors by proper adjustment of K and K_t.

SECTION II - THE METHOD FOR GETTING A PARABOLIC LINEAR ZONE.

The equation of the parabolic linear zone is $X + K_T \dot{X}^2 = 0$ Then by passing the tachometer feedback through a squaring tube and adjusting the gain, the linear zone can be made a parabola. Some of the squaring tube circuits are in the book "Electron-Tube Circuits" by Samuel Seely.

The result of using a parabolic linear zone in a discontinuous feedback system will be much better than that of using a linear zone with straight line boundaries, especially for the small signal. If straight line segments are used to approximate the parabola, then the circuit in Fig. 3-11 is a practical one.

SECTION III - DATA AND FIGURES

A. Saturated system without tachometer feedback

(a) $G(S) = \dfrac{\omega_n^2}{S^2} = \dfrac{1}{S^2}$ (Fig. 3-2a)

$N = \dfrac{d\dot{E}}{dE} = \dfrac{\pm 1}{E}$ For saturated region

N	-0.5	-1	-0.333	-0.25
$\dot{E}$	2	1	3	4

$N = -\dfrac{E}{N}$ For linear zone.

(b) $G(S) = \dfrac{A}{S(S+a)} = \dfrac{1}{S(S+0.23)}$ (Fig. 3-2b)

$N + 0.23 = \pm \dfrac{1}{E}$ For saturated region

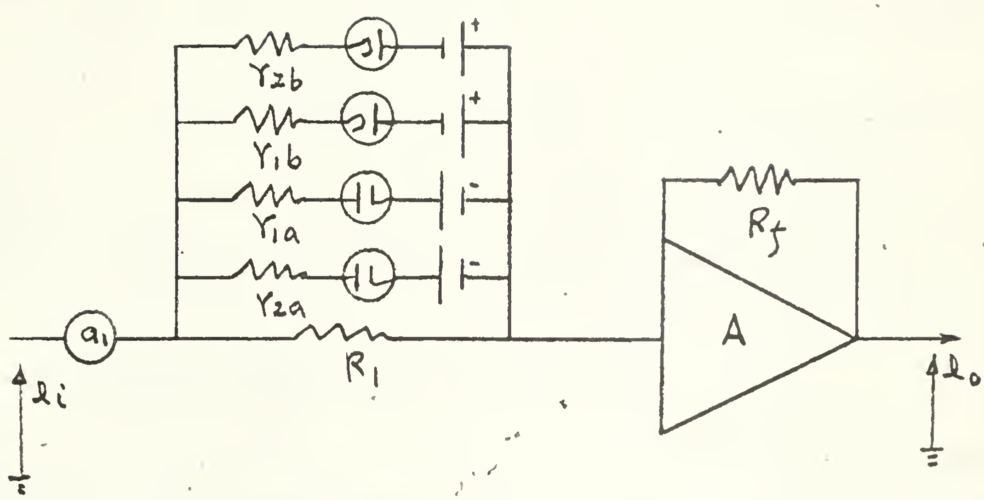

Fig. 3-11 A Practical Circuit for Producing Parabolic Linear Zone.

N	-1	-0.71	-0.503	-0.5	-0.46
$\dot{E}$	-1.3	-2.08	-3	-3.7	-4.35

$$\dot{E} = \frac{-E}{N + 0.23} \qquad \text{For linear zone.}$$

B. The effect of tachometer feedback on the phase portrait

(a) $$\ddot{E} + K_T\dot{E} + KE = 0 \qquad \text{(Fig. 3-3a)}$$

$$\ddot{E} + 0.23\dot{E} + E = 0$$

$$\dot{E} = \frac{-E}{N + 0.23} = KE \qquad \text{For linear zone}$$

N	0	1	-1	-0.25	-0.333	-0.25	-0.23
K	-4.35	-0.814	1.3	3.7	1.77	2.28	∞

Slope of linear zone $-\frac{1}{K_T} = -4.35$

$$\dot{E} = \frac{-1}{N} \qquad \text{For saturated region}$$

(b) Critically damped case. $G(s) = \frac{1}{s^2}$ slope of the linear zone is half of the slope of eigenvector. (Fig. 3-3b).

(c) Slope of linear zone is less than that of the slow eigenvector.

$$\ddot{E} + (a + K_T)\dot{E} + E = 0$$

where $K_T = 1.43$, $a = 0.62$ (Fig. 3-5b)

$$\ddot{E} + 2.05\dot{E} + E = 0$$

For saturated region $\dot{E} = \frac{-1}{N + 0.62}$

N	0	1	2	0.5	0.212
$\dot{E}$	-1.61	-0.618	-0.383	-0.892	-1.2

(d) Slow eigenvector and linear zone parallel. (Fig. 3-5a)

By using $K_T = 1.5$, $a = 0.668$

Then $\ddot{E} + 2.168\dot{E} + E = 0$, $r_1 = -1.5$, $r_2 = -0.668$

For linear operation

$$\dot{E} = \frac{-E}{N + 2.168} = KE$$

N	0	1	−0.168
K	−0.461	−0.316	−0.5

For saturated region

$$\dot{E} = \frac{-1}{N + 0.668}$$

N	0	1.532	0.588	0.332	0.947	0.167
$\dot{E}$	−1.5	−0.4	−0.8	−1.0	−1.4	−1.2

(e) Fast eigenvector and linear zone parallel. (Fig. 3-12)

For $K_T = 0.668$, $a = 1.5$

$$\ddot{E} + 2.168\dot{E} + E = 0 \qquad r_1 = 1.5,\ r_2 = 0.668$$

For linear zone, same as in Fig.3-3d

For saturated region

$$\dot{E} = \frac{-1}{N + 1.5}$$

N	0	1
$\dot{E}$	−0.668	−0.4

(f) Linear zone has a greater slope than the fast eigenvector

(Fig. 3-13)

For $K_T = 0.368$, $a = 1.8$

$$\ddot{E} + 2.168\dot{E} + E = 0$$

$$r_1 = -1.5, \quad r_2 = -0.668$$

Saturated velocity $\dot{E}_S = \frac{1}{1.8} = 0.555$

For saturated region

$$\dot{E} = \frac{-1}{N + a} = \frac{-1}{N + 1.8}$$

N	0.7	3.2
$\dot{E}$	−0.4	−0.2

C. Effect of τ on the slope of dieout trajectory. (Fig. 3-4)

$$G(s) = \frac{16.62}{S(\tau S + 1)}$$

$$\dot{E} = \frac{16.62}{N + 1} \qquad \text{For } \tau = 1$$

N	0	−1	−2	−5	−10
$\dot{E}$	16.62	∞	−16.62	−4.16	−1.85

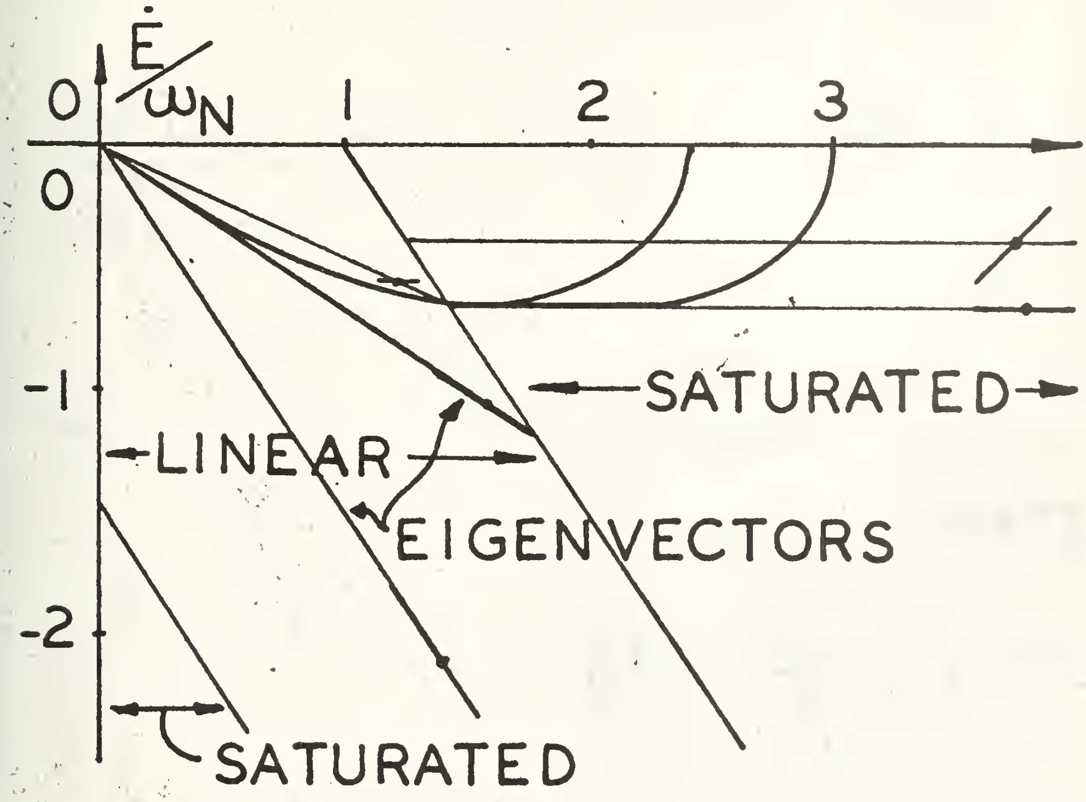

Fig. 3-12 Fast Eigenvector and Linear Zone Parallel

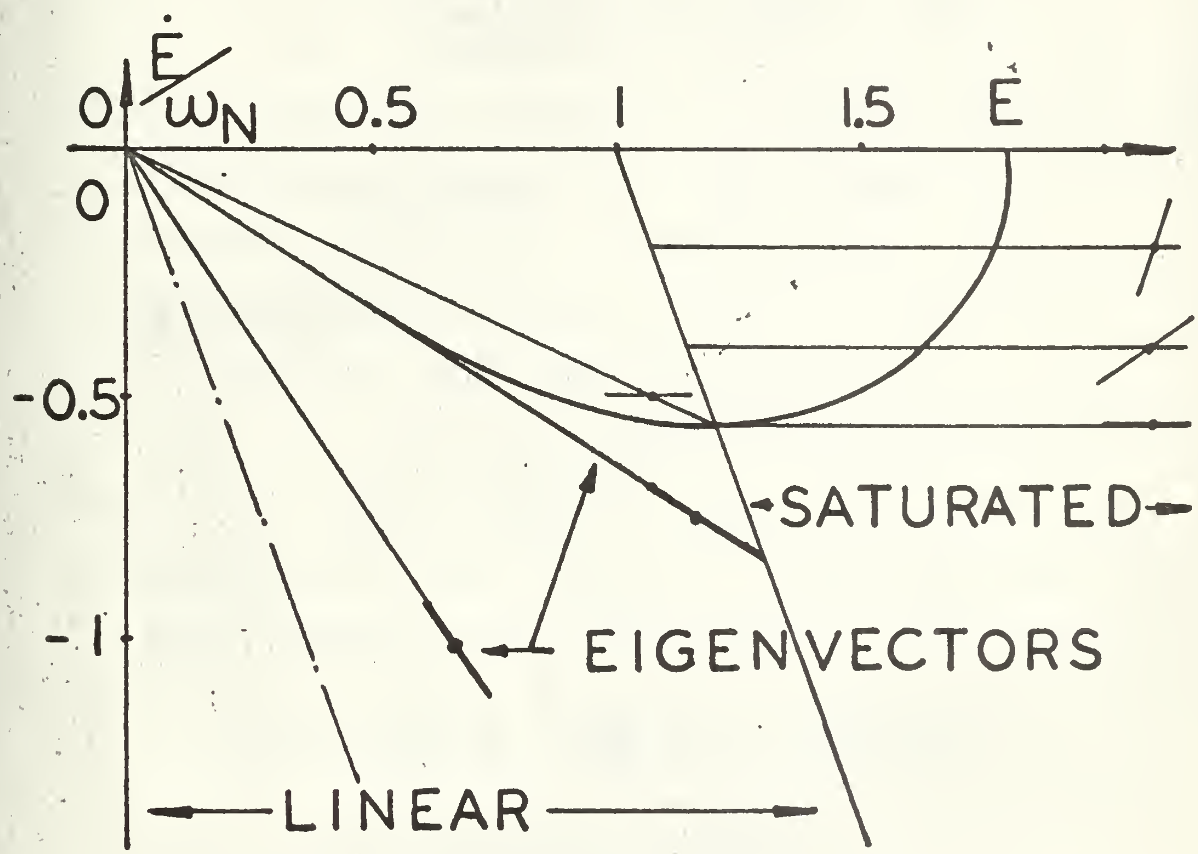

Fig. 3-13 Linear Zone Has a Greater Slope Than the Fast Eigenvector.

For $\tau = 0.2$, $\dot{E} = \dfrac{83.3}{N+5}$

N	0	-5	-10	-20
E	16.62	∞	-16.62	-5.56

For $\tau = 1.662$, $\dot{E} = \dfrac{10}{N+0.6}$

N	0	-0.6	-1.032	-2.27
E	16.62	∞	-10	- 5

D. Analog computer study of discontinuously damped systems (Fig. 3-10a)

$$G(S) = \frac{1}{S(S+31.62)}, \qquad K_T = 0.088$$

Saturating amplifier gain K = 1000

Saturated output of amplifier = 12.5^{V}

$$\alpha_t = 20, \qquad \alpha_{\theta_R} = \alpha_{\theta_c} = \alpha_K = 0.01$$

$$\alpha_J = 0.1$$

$$\alpha_w = \frac{0.1166}{7.4} = 0.0158$$

Relay open at $\overline{\theta_R} = 1.25^{V}$, i.e. $\theta_R = 0.0125$ rad.

Relay close (drop out) at $\overline{\theta_R} = 0.4^{V}$

$$[-\overline{J}] = K\left[\frac{\alpha_{\theta_R}}{\alpha_J}\overline{\theta_R} - \frac{\alpha_{\theta_c}}{\alpha_J}\overline{\theta_c} - K_T\frac{\alpha_{\dot{\theta}_c}}{\alpha_J}\overline{\dot{\theta}_c}\right]$$

$$= 100\,\overline{\theta_R} - 100\,\overline{\theta_c} - 13.9\,\overline{\dot{\theta}_c}$$

$$\overline{\omega_c} = \overline{\dot{\theta}_c} = \frac{\dfrac{\alpha_J}{\alpha_w\alpha_t}[-\overline{J}]}{P + \dfrac{31.62}{\alpha_t}}$$

$$= \frac{0.316}{P+1.58}[-\overline{J}]$$

$$[-\overline{\theta_c}] = \frac{1}{P}\left[\frac{\alpha_{w_c}}{\alpha_{\theta_c}\alpha_t}\overline{\omega_c}\right] = \frac{1}{P}[0.079\,\overline{\omega_c}]$$

E. Calibration and parameter evaluation for physical system.

(a) Error pot calibration in Fig. 3-14 is 6.705volts/rad.

(b) Tachometer calibration is 15.15^{V} /rad/sec (By using stop watch and counting the number of rotations of output shaft.)

(c) Saturating amplifier gain $K_A = 440$. (By comparing the output and input voltage of the amplifier in the linear region.)

(d) Find the approximate natural frequency by operating the system without feedback $\omega_n = 15.7$ rad./Sec.

(e) From the x - y plot (E vs $\dot{E}$) find the slope of linear zone, and find the tachometer feedback. $\frac{1}{K_T} = 7.6$, $K_T = 0.1315$

(f) Find the motor load gain K_m by the relation

$$\omega_n^2 = K_A K_m = 247$$

Then $K_m = 0.56$

(g) From E vs $\dot{E}$ plot find the saturated velocity ω_{sat}.

$$\omega_{sat.} = \frac{K_m}{a} = 2.97 \quad \text{rad./sec.}$$

Then $a = 14.15$

The final result is

$$\ddot{E} + (a + K_A K_m K_T)\,\dot{E} + \omega_n^2\, E = 0$$

$$\ddot{E} + (14.15 + 247 \times 0.132)\dot{E} + 247\, E = 0$$

$$\ddot{E} + 46.65\ \dot{E} + 247 = 0$$

$$r_1 = -6.175, \quad r_2 = -40.475$$

Note: The frequency response method cannot be used here, because the stiction of this system is very large, and the superposition theory is not valid. By increasing the load inertia, the step response character of this 2nd order physical system is recorded in Fig. 3-15.

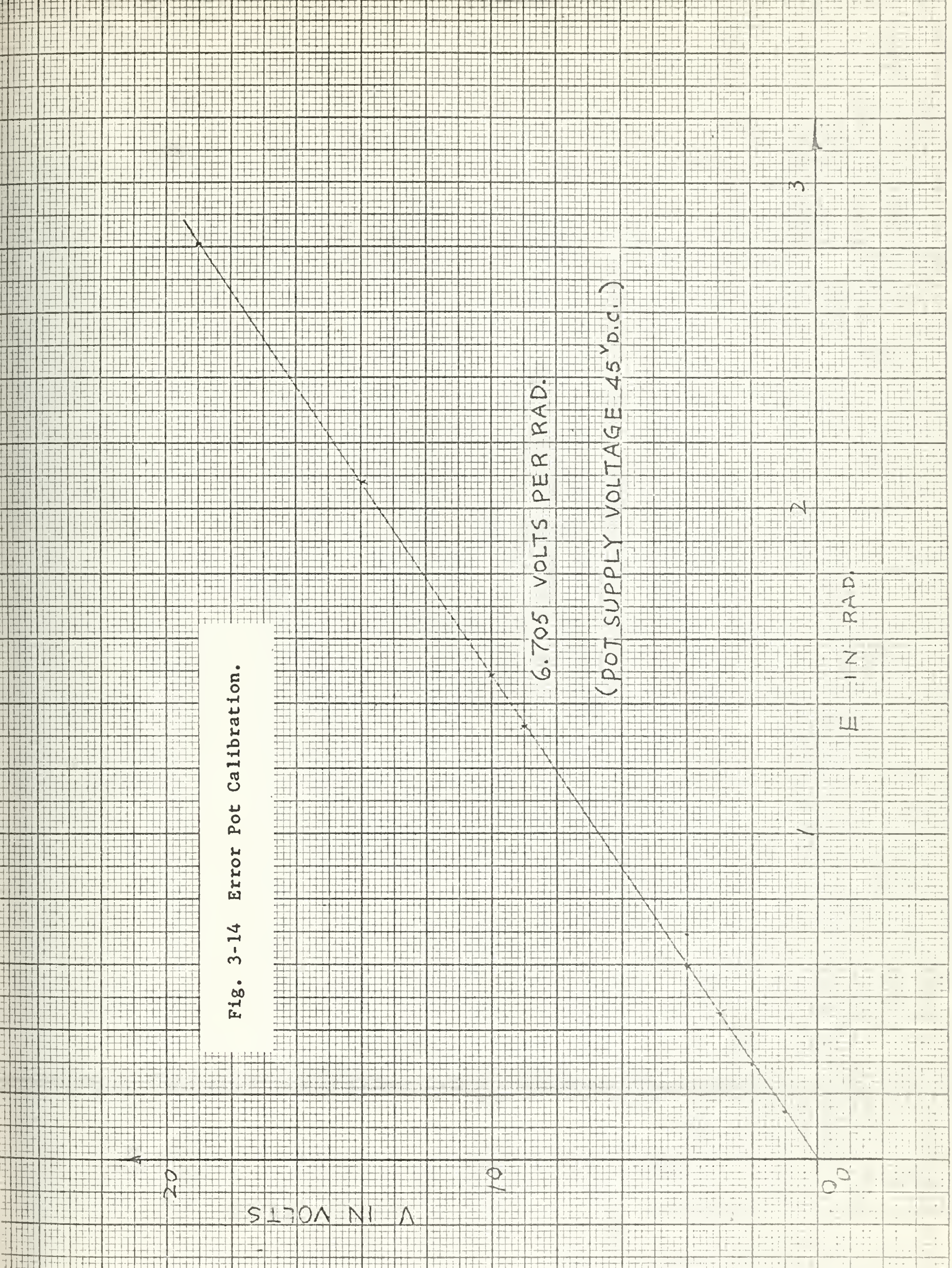

Fig. 3-14 Error Pot Calibration.

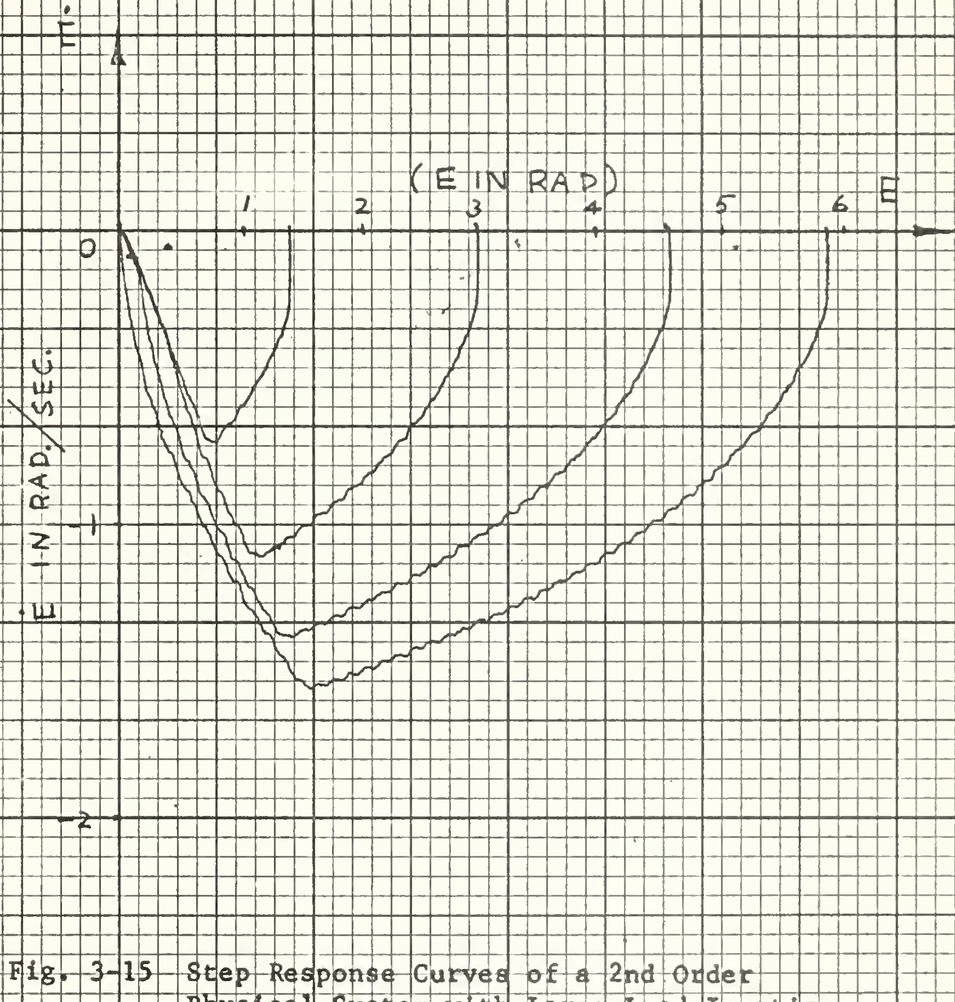

Fig. 3-15 Step Response Curves of a 2nd Order Physical System with Large Load Inertia.

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2. Analysis and Design Principles of Second and Higher Order Saturating Servomechanisms, R. E. Kalman. Trans AIEE, Pt. II, Nov. 1955.

3. Phase Space Analysis and Design of Linear Discontinuously Damped Feedback Control Systems. K. W. Han and G. J. Thaler. AIEE Paper, CP 60-1021.

CHAPTER IV - ANALYSIS AND DESIGN OF SEMILINEAR AND SATURATED HIGH ORDER SYSTEMS

Part I - GENERAL DESCRIPTION

SUMMARY

An investigation is made to determine the relative position between linear space and eigenspaces in the nth order error space for the purpose of making an nth order feedback control system semilinear or with fully saturated operation so that fast response is obtained without a complex computer and relay device.

DEFINITIONS

EIGENPLANE: The plane corresponding to an eigenvector, in third order error space it is a two dimensional plane, in nth order space, it is a subspace with order N-1.

LINEAR PLANARY SPACE: A linear space in 3rd order error space, it is a space between two parallel planes formed by the feedback signals.

AXISAL PLANE: The center plane of the linear planary space.

SEMILINEAR SYSTEM: For a step input the first part of the trajectory in the error space is saturated and the second part is linear.

SATURATED SYSTEM: The trajectory of the system is saturated in both rising and die-out part for a step input in error space.

SECTION I - INTRODUCTION

The optimum servo systems obtained by means of nonlinear and switching techniques, have received considerable attention in recent years. On the other hand for linear systems the switching surface corresponding to a fast eigenvector has also been proved to be useful. By using phase space analysis techniques it was found that the linear space can be controlled by feedback signals and also the slow eigenvector tends to control the system in the linear space. A method is introduced to calculate the proper values of feedback signals and the main gain of the system, to make the slow eigenvector space combine with the linear space, or make the slow eigenspace have larger (more negative) slope than the linear space. The former gives a semi-linear system in which all the trajectories after coming out from the saturated region will always stay in the linear space and go to the origin along the slow eigenspace. The latter gives a nearly fully saturated system, the linear space forming the switching action and the slow eigenspace drives the system saturated in the opposite direction, then a system has nearly optimum response. For this case the system is using full power all the time.

SECTION II - SEMILINEAR SYSTEMS.

For a nth order feedback control system as in Fig. 4-1, the characteristic equations are

$$\overset{n}{E} + A\overset{n-1}{E} + B\overset{n-2}{E} + \cdots\cdots K E = 0 \qquad \text{(without compensation)} \quad (4\text{-}1)$$

$$\overset{n}{E} + (KK_1 + A)\overset{n-1}{E} + (Kk_2 + B)\overset{n-2}{E} + \cdots\cdots K E = 0 \qquad (4\text{-}2)$$

(with compensating feedbacks all to error input)

The roots of equation (4-2) are $r_1, r_2, \cdots\cdots r_n$, then

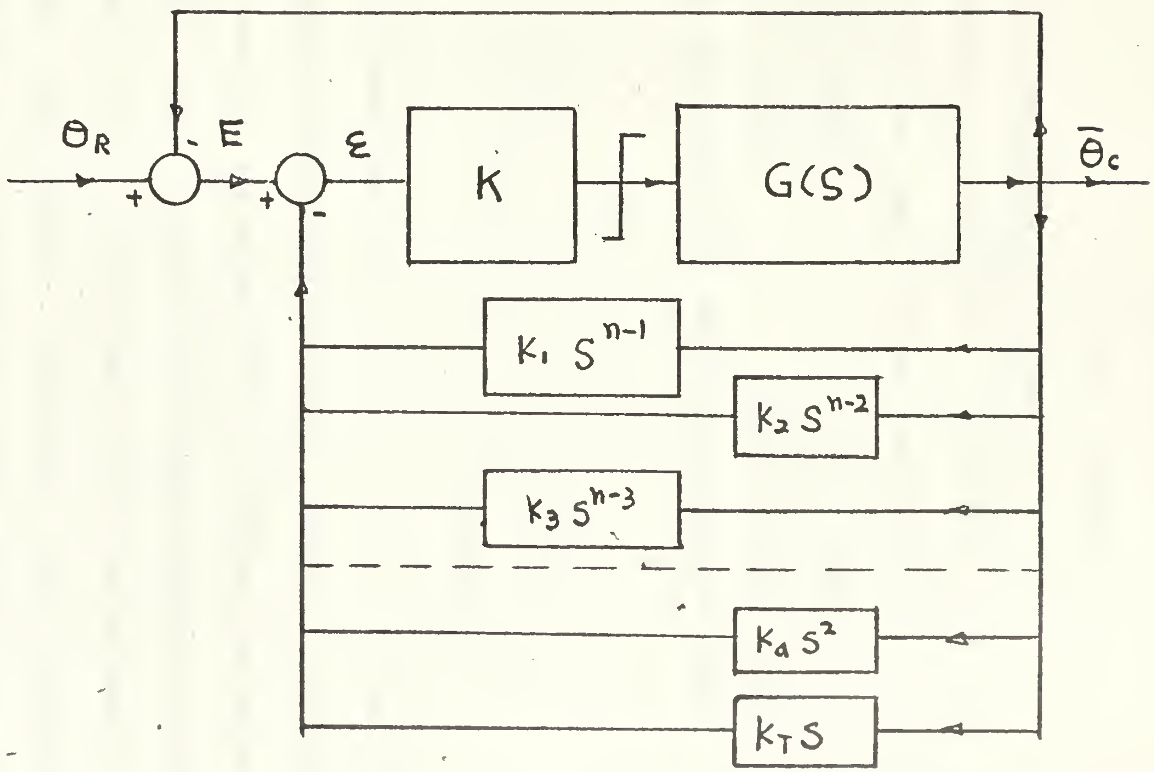


Fig. 4-1 Block Diagram of a Nth Order Feedback Control System with Saturated Main Amplifier

$$\left.\begin{aligned}
&r_1 + r_2 + r_3 + - - - - = K K_1 + A \\
&r_1 r_2 + r_1 r_3 + - - - r_1 r_n + r_2 r_3 + r_2 r_4 + - - - - r_n r_{n-1} = K K_2 + B \\
&- - - - - - - - - - - - \\
&- - - - - - - - - - - - \\
&r_1 r_2 - - - - r_n = 1
\end{aligned}\right\} \tag{4-3}$$

Equation for slow eigenvector (space) is

$$\overset{n-1}{E} + (r_1 + r_2 + - - - - r_{n-1}) \overset{n-2}{E} + - - - (r_1 r_2 + r_1 r_3 + - - - r_{n-2} r_{n-1}) \dot{E} - r_1 - - - r_{n-1} E = 0 \tag{4-4}$$

Equation for linear space is (just like in 2nd order case decided by K_t) at

$$K_1 \overset{n-1}{E} + K_2 \overset{n-2}{E} + - - - - - - - - K_a \ddot{E} + K_T \dot{E} + E = 0 \tag{4-5}$$

or

$$\overset{n-1}{E} + \frac{K_2}{K_1} \overset{n-2}{E} + - - - - - - - - \frac{K_a}{K_1} \ddot{E} + \frac{K_T}{K_1} \dot{E} + \frac{1}{K_1} E = 0 \tag{4-6}$$

If equation (4-4) and (4-6) define the same space then

$$\left.\begin{aligned}
K_1 &= \frac{1}{r_1 r_2 - - - - r_{n-1}} \\
K_2 &= (r_1 + r_2 + r_3 + - - + r_{n-1}) K_1 \\
K_3 &= (r_1 r_2 + r_1 r_3 + - - - - + r_{n-2} r_{n-1}) K_1 \\
&- - - - - - - - - - - - -
\end{aligned}\right\} \tag{4-7}$$

For a system given A, B, -----------------K, the values of the roots can be found by solving equation (4-3) and (4-7), and then substitute the value of roots into equation (4-7) to calculate the proper value of each coefficient. The thickness of the linear space can be adjusted by the main channel gain, because of the slope of the slow eigenspace combined with the linear space, the system will always keep linear operation at the final part of the trajectory, no matter how much disturbance or how many nonlinearities were encountered in the former part of the trajectory. To obtain fast response the former part is always operated at the saturated condition. Then it is called a semilinear system. This is just as in the second order case, a slow eigenvector is combined with the axis of the linear zone.

SECTION III - THIRD ORDER EXAMPLE FOR SEMILINEAR SYSTEM.

Assume the original system characteristic equation is

$$\dddot{E} + 6\ddot{E} + 8\dot{E} + 64 E = 0 \tag{4-8a}$$

The characteristic equation after adding feedback signals is

$$\dddot{E} + (64K_a + 6)\ddot{E} + (64K_T + 8)\dot{E} + 64 E = 0 \tag{4-8b}$$

From equation (4-3)

$$\gamma_1 + \gamma_2 + \gamma_3 = 64 K_a + 6 \tag{4-9}$$

$$\gamma_1\gamma_2 + \gamma_2\gamma_3 + \gamma_1\gamma_3 = 64 K_T + 8 \tag{4-10}$$

$$\gamma_1\gamma_2\gamma_3 = 64 = K \tag{4-11}$$

From (4-7) $\dfrac{1}{K_1\gamma_2} = K_a \cdots (4-12)$, $K_a(\gamma_1 + \gamma_2) = K_T$ (4-13)

Since $K = \gamma_1\gamma_2\gamma_3 = 64$, $K_a = \dfrac{1}{\gamma_1\gamma_2}$ then from (4-9) and (4-10)

$$\gamma_1 + \gamma_2 = A = 6 \tag{4-14}$$

$$\gamma_1\gamma_2 = B = 8 \tag{4-15}$$

The values of the roots are $\gamma_1 = 2$, $\gamma_2 = 4$, $\gamma_3 = 8$

From equation (4-12)

$$K_a = \frac{1}{8}, \quad K_T = \frac{6}{8} \tag{4-16}$$

The equation of the linear planary space is

$$K_a\ddot{E} + K_T\dot{E} + E = 0 \tag{4-17}$$

$$\ddot{E} + 6\dot{E} + 8E = 0 \tag{4-18}$$

The equation of the slow eigenvector is

$$\ddot{E} + (2+4)\dot{E} + 2\times 4 E = 0 \tag{4-19}$$

$$\ddot{E} + 6\dot{E} + 8E = 0 \tag{4-20}$$

In this example if $K = 24$ then $\gamma_1 = 2$, $\gamma_2 = 4$, $\gamma_3 = 3$.

Then the eigenplane corresponding to the root with middle value γ_3 is combined with the linear planary space, i.e., the equation of the axisal plane of the linear planary space is

$$\ddot{E} + 6\dot{E} + 8E = 0 \tag{4-21}$$

and the equation of the eigenplane is also

$$\ddot{E} + 6\dot{E} + 8E = 0 \qquad (4\text{-}22)$$

Again if the gain is K = 12, then $r_1 = 2$, $r_2 = 4$, $r_3 = 1.5$.

The fast eigenplane combined with the axisal plane, the equation for eigen-plane and axisal plane is the same as equation (4-21).

5th order example :

Assume that the open loop transfer function is

$$G(s) = \frac{K}{s(s+p_1)(s+p_2)(s+p_3)(s+p_4)} \qquad (4\text{-}23)$$

and let $K = 216$, $p_1 = 1.5$, $p_2 = 2$, $p_3 = 3$, $p_4 = 4$

Then the characteristic equation is

$$\overset{.....}{E} + 10.5\overset{....}{E} + 39.5\dddot{E} + 63\ddot{E} + 36\dot{E} + 216E = 0 \qquad (4\text{-}24)$$

(without feedback)

or $$\overset{.....}{E} + (kk_1 + 10.5)\overset{....}{E} + (kk_2 + 39.5)\dddot{E} + (kk_a + 63)\ddot{E} + (kk_T + 36\dot{E}) + 216E = 0 \qquad (4\text{-}25)$$

(with feedback)

Where k_1, k_2, k_a, k_T, are coefficients of feedback signals. For a semilinear system, from Equation (4-7)

$$\frac{1}{r_1 r_2 r_3 r_4} = k_1 \qquad (4\text{-}26)$$

$$(r_1 + r_2 + r_3 + r_4)k_1 = k_2 \qquad (4\text{-}27)$$

$$(r_1 r_2 + r_2 r_3 + r_3 r_4 + r_1 r_3 + r_1 r_4 + r_2 r_4)k_1 = k_a \qquad (4\text{-}28)$$

$$(r_1 r_2 r_3 + r_1 r_3 r_4 + r_2 r_3 r_4 + r_1 r_2 r_4)k_1 = k_T \qquad (4\text{-}29)$$

From equation (4-3)

$$r_1 + r_2 + r_3 + r_4 + r_5 = kk_1 + 10.5 \qquad (4\text{-}30)$$

$$(r_1 r_2 + r_1 r_3 + r_1 r_4 + \cdots r_4 r_5) = kk_2 + 39.5 \qquad (4\text{-}31)$$

$$(r_1 r_2 r_3 + r_1 r_3 r_4 + \cdots r_3 r_4 r_5) = kk_a + 63 \qquad (4\text{-}32)$$

$$(r_1 r_2 r_3 r_4 + \cdots r_2 r_3 r_4 r_5) = kk_T + 36 \qquad (4\text{-}33)$$

$$r_1 r_2 r_3 r_4 r_5 = K \qquad (4\text{-}34)$$

From equation (4-26), (4-30) and (4-34)

$$\left.\begin{aligned} r_1 + r_2 + r_3 + r_4 &= 10.5 \\ r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 &= 39.5 \\ r_1r_2r_3 + r_1r_2r_4 + r_2r_3r_4 + r_1r_3r_4 &= 63 \\ r_1r_2r_3r_4 &= 36 \end{aligned}\right\} \qquad (4\text{-}35)$$

Solving equation (4-35): $r_1 = 1.5,\ r_2 = 2,\ r_3 = 3,\ r_4 = 4,\ r_5 = 6.$

Note that all the values of roots except the 5th root are decided by the open loop poles, **but** not **a**ffected by the system gain K. Then the gain K only changes the thickness of the linear space, but not the slope of it. The slope of the linear space is decided by the other four roots.

The equation of the slow eige**nspace is**

$$\overset{\cdot\cdot\cdot\cdot}{E} + (r_1+r_2+r_3+r_4)\dddot{E} + (r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4)\ddot{E}$$

$$+(r_1r_2r_3+r_1r_2r_4+r_2r_3r_4+r_1r_3r_4)\dot{E} + r_1r_2r_3r_4 E = 0 \qquad (4\text{-}36)$$

$$\overset{\cdot\cdot\cdot\cdot}{E} + 10\dddot{E} + 39.5\ddot{E} + 63\dot{E} + 36E = 0 \qquad (4\text{-}37)$$

The coefficients in this equation correspond **to those** of the original **under**damped characteristic equation with the order decreased by one. That is the coefficient of $\overset{h}{E}$ term is equal to the coefficient of $\overset{h+1}{E}$ term in the **underdamped** characteristic equation.

From equation (4-6)

$$\overset{\cdot\cdot\cdot\cdot}{E} + \frac{K_2}{K_1}\dddot{E} + \frac{K_a}{K_1}\ddot{E} + \frac{K_T}{K_1}\dot{E} + \frac{1}{K_1}E = 0 \qquad (4\text{-}38)$$

From equation (4-37), (4-38)

$$K_1 = \frac{1}{36}$$

$$K_T = 63 \times \frac{1}{36}$$

$$K_a = 39.5 \times \frac{1}{36}$$

$$K_2 = 10.5 \times \frac{1}{36}$$

Then the coefficients of feedback signals are decided.

For $r_5 = 6$, then

$$k = r_1r_2r_3r_4r_5 = 216$$

The characteristic equation of the compensated system is

$$E^{(5)} + \left(216\times\frac{1}{36}+10.5\right)E^{(4)} + \left(216\times 10.5\times\frac{1}{36}+39.5\right)\dddot{E}$$

$$+\left(216\times 39.5\times\frac{1}{36}+63\right)\ddot{E} + \left(216\times 63\times\frac{1}{36}+36\right)\dot{E} + 216E = 0$$

i.e. $$E^{(5)} + 16.5E^{(4)} + 102.5\dddot{E} + 300\ddot{E} + 414\dot{E} + 216E = 0 \qquad (4\text{-}39)$$

SECTION IV - DISCUSSIONS AND DESIGN CONSIDERATIONS

(A) <u>The discussion about normal critical case and other cases</u>: From Eq. (4-3) in the second term (of the semilinear system) the coefficient of feedback signal times K is equal to the nth root of the compensated characteristic equation. Such as in Eq. (4-3) and Eq. (4-4).

$$K K_1 = r_n$$

and for the third term $\quad K K_2 = r_n (r_1 + r_2 + \cdots + r_{n-1})$

In a particular case that the real roots are all equal, call this a NORMAL CRITICAL CASE. But in other cases when the system has the same equal real roots it may not have the same character as that of normal critical case, because the linear space plays an important role in this semilinear system. For example, in two third order systems the three real roots are the same, $(r_1 = r_2 = r_3 = 4)$ for both these two systems after compensation. The characteristic equations before compensation are $\dddot{E} + 8\ddot{E} + 16\dot{E} + 64E = 0$ and $\dddot{E} + 3\ddot{E} + 10\dot{E} + 64E = 0$, after compensation are:

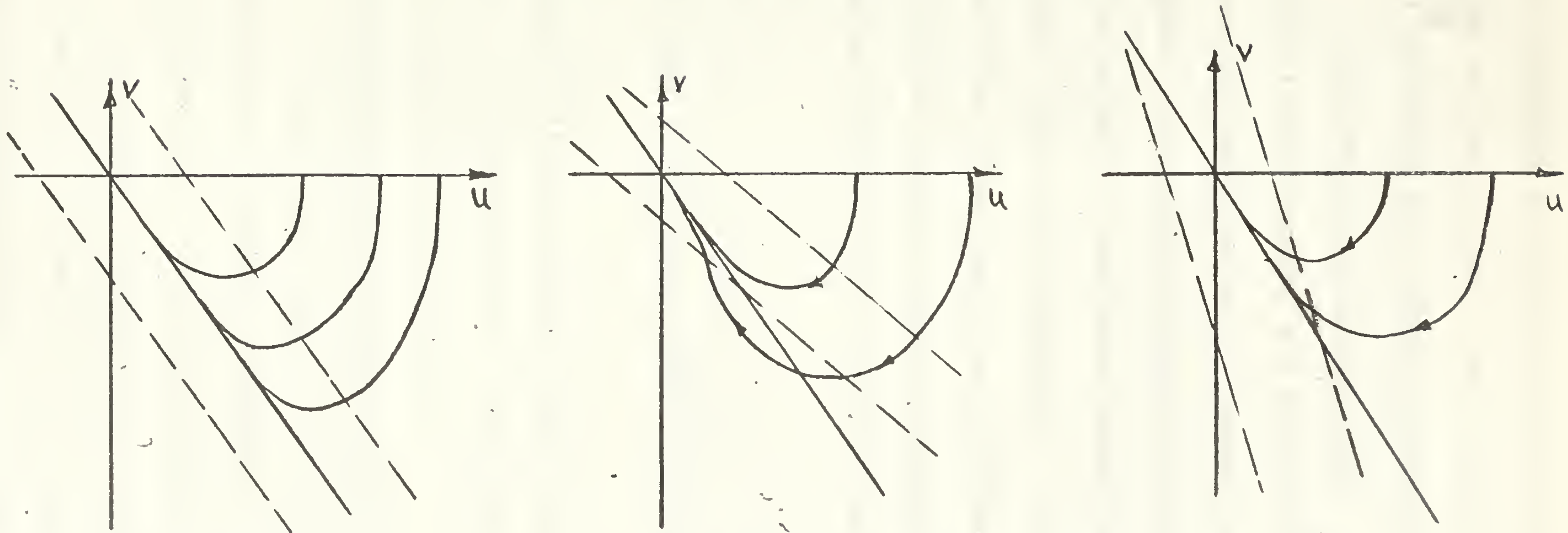
$$\dddot{E} + (K K_a + 8)\ddot{E} + (K K_t + 16)\dot{E} + 64E = 0 \qquad (4\text{-}40)$$

$$\dddot{E} + (K K_a' + 3)\ddot{E} + (K K_t' + 10)\dot{E} + 64E = 0 \qquad (4\text{-}41)$$

Equation (4-40) is a normal critical case, it can never be saturated in the opposite direction. It is a linear or semilinear system as sketched in Fig. 4-2a. From Eq. (4-41) it is known that the slope of the eigenplane is different from the axisal plane. If the gain is high enough to cause saturation, then the system will be saturated in the opposite direction, because the slope of the slow eigenplane (here only one eigenplane) is larger (more negative) than the axisal plane. The typical trajectory is sketched in Fig. 4-2b.

In the other case, if the characteristic equation is

$$\dddot{E} + (K K_a'' + 10)\ddot{E} + (K K_t'' + 30)\dot{E} + 64E = 0 \qquad (4\text{-}42)$$



(A) (B) (C)

Fig. 4-2 Sketched trajectories for critically damped systems.
(a) Normal critical case
(b) Eigenvector has larger (more negative) slope than linear planary space.
(c) Eigenvector has smaller slope.
(where u, v are the coordinate axes for the transformed E vs $\dot{E}$ plane).

then the difference is not much since no trajectory can pass the slow eigenplane in linear region. The trajectory is sketched in Fig. 4-2c.

(B) Discussion about linear planary space combined with fast eigenvector and the use of switching:

In the low gain system, the system has a very wide linear space. Then the continuous semilinear and saturated systems are not suitable to make the system have fast response, because in a semilinear system the slow eigenspace is used to control the continuous damped system. The fast response is due to the small motor load time constant. In the saturated system the fast response is due to the high gain, which dirves the slow eigenspace clockwise, then use of the saturated die-out trajectory provides optimum response. Both of these two systems require a narrow linear space, in order to apply full voltage.

But in case the system gain cannot be high, the linear space becomes predominant, then to design the system as a linear system seems to be more practical.

Since the linear switching methods have been investigated in Chapter II then here it is only necessary to check the equation of fast eigenplane to see if it stays in the linear space.

(C) Discussion for semilinear system and design procedure.

(1) Semilinear system has the particular character that the main amplifier gain can be changed independently, then it can give the desired amount of saturation. The design procedure is quite simple.

(2) Because, for this kind of system the slope of the slow eigenspace is decided by the values of the open loop poles, the response is independent of the gain adjustment. The larger the pole values the faster the response, because the increase of gain can change the slopes of the eigenspaces

but not the slowest one.

(3) The benefit of this kind of system is that (a) the full power is applied in the saturated part of the trajectory; (b) the system will never have overshoot no matter what kind or how large the input signal is; (c) It is easy to design and adjust.

(4) Design procedures:

(a) From the poles of the original open loop transfer function write down the equation of both linear space and slow eigenspace.

(b) Select maximum permissible gain to decide the nth root of the compensated characteristic equation.

(c) From equation (4-6) find all the coefficients of feedback signals.

SECTION V - SATURATED SYSTEM

When the slope of the slow eigenspace is larger than the slope of linear space, then the trajectory will be saturated in the opposite direction after passing through the linear space. This can be done by increasing the main channel gain or/and the gain of the feedback signals. The characteristic equation of the compensated system can be written as

$$\overset{n}{E} + (K'K' + A)\overset{n-1}{E} + (K'K_2' + B)\overset{n-2}{E} + \cdots + K'E = 0 \qquad (4\text{-}43)$$

where K's are larger than the K's in the semilinear system. Assume the roots of equation (4-43) are $r_1', r_2', \cdots r_n'$, they are all larger than the roots of the original semilinear system. Then the equation of the slow eigenspace is

$$\overset{n-1}{E} + (r_1' + r_2' + \cdots r_{n-1}')\overset{n-2}{E} + (r_1'r_2' + r_1'r_3' + \cdots r_{n-2}'r_{n-1}')\overset{n-3}{E} + \cdots r_1 r_2 \cdots r_n E = 0 \qquad (4\text{-}44)$$

and the equation of linear space is

$$K_1'\overset{n-1}{E} + K_2'\overset{n-2}{E} + \cdots + K_4'\dddot{E} + K_T'\dot{E} + E = 0 \qquad (4\text{-}45)$$

or

$$\overset{n-1}{E} + \frac{K_2'}{K_1'}\overset{n-2}{E} + \cdots + \frac{K_a'}{K_1'}\ddot{E} + \frac{K_T'}{K_1'}\dot{E} + \frac{E}{K_1'} = 0 \qquad (4\text{-}46)$$

Since the slope of the linear space is decreased (less negative) as the gain of the feedback signal is increased, the slope of this linear space is less negative than that of the original semilinear system. Therefore, the slope of the slow eigenspace is clearly larger(more negative) than the slope of the linear space, and a saturated system is thus defined providing the gain is large enough to cause saturation. Since the only condition is to make the slope of the slow eigenspace larger than the slope of linear space, then the relation between the gain of feedback signals and the values of the real roots are not critical. The main points of designing such a system are to make sure the form of the die-out trajectory, the maximum input and the proper slope of linear space to act as a switching space.

Since the trajectory in the saturated region is decided by the uncompensated conditions, then the feedback signal adjustment has no effect upon the

shape of the trajectory in the die-out part in a saturated system. However considering the response for small signal input that causes no saturation in any part of the trajectory, then a critically damped case is preferable provided that the roots are larger than the roots of semilinear case for this same system.

SECTION VI - INITIAL CONDITION AND FEEDBACK SIGNAL CONSIDERATION

The slow eigenspace divides the whole error space into two parts, just as the slow eigenvector divides the phase plane into two parts. On the other hand, the linear space also divides the whole error space into two parts. The trajectory starts due to any initial conditions and ends tangent to the slow eigenspace. For example in a third order system if plane is constructed perpendicular to the linear planary space, then the trajectories due to various initial conditions will be like those in Fig. 4-3 .

The intersection of the linear space and slow eigenspace need not be in the $\dot{E}$ vs $\ddot{E}$ plane, because some of the trajectories may start from the positive half error space and go to the negative half error space and then go to the origin.

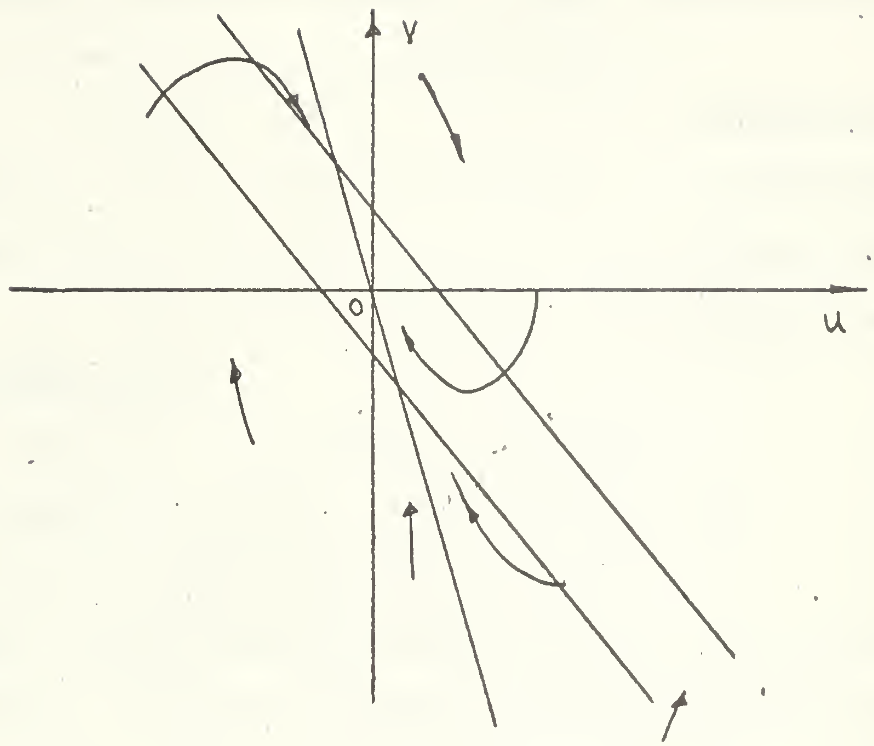
Since there are n-1 derivative feedback signals for an nth order system any one of them can cause saturation. For so called bang-bang systems using feedback signals to make the main amplifier saturate, the design consideration must include all the feedback signals. These high order feedback signals in physical systems are not easy to obtain, or are too noisy to be used as control signals. Therefore, if the saturation character of the system can be decided (controlled) only by low order signals, this kind of saturated system will be more useful.

For any system if the characteristic equation can be written as

$$\overset{n}{E} + A\overset{n-1}{E} + B\overset{n-2}{E} - \cdots + KE = 0 \qquad \text{(uncompensated)} \qquad (4\text{-}47)$$

$$\overset{n}{E} + (KK_1 + A)\overset{n-1}{E} + (KK_2 + B)\overset{n-2}{E} + \cdots + KE = 0 \qquad \text{(compensated)} \qquad (4\text{-}48)$$

and from the analysis used before, the semilinear equation can be easily written down provided the gain is decided. The condition for any one of these feedback signals to cause saturation is that the magnitude of that feedback signal has to be larger than the magnitude for the semilinear case.



Linear Planary Space

Fig. 4-3 A Sketch of the Trajectories due to Various Initial Conditions

On the other hand, if the set of real roots selected for the saturated system can be arranged to make some of the coefficients the same as the semilinear system (overdamped), then the feedback signals corresponding to these unchanged coefficients will never be saturated, because for them the semilinear condition still holds.

It is usually desired to eliminate the highest derivative feedback signal. For example, in the third order system used before

$$\dddot{E} + 6\ddot{E} + 8\dot{E} + 64E = 0 \qquad \text{(uncompensated)} \quad (4\text{-}49)$$

$$\dddot{E} + (kk_a + 6)\ddot{E} + (kk_T + 8)\dot{E} + 64E = 0 \qquad \text{(compensated)} \quad (4\text{-}50)$$

For the linear case $r_1 = 2,\ r_2 = 4,\ r_3 = 8$ (compensated)

Here use: $r_1' = 3,\ r_2' = 3,\ r_3' = 7.1$ the compensated characteristic equation will be:

$$\dddot{E} + (7.1 + 6)\ddot{E} + (43.6 + 8)\dot{E} + 64E = 0 \quad (4\text{-}51)$$

compare with semilinear case for $r_1 = 2,\ r_2 = 4,\ r_3 = 8$

$$\dddot{E} + (8 + 6)\ddot{E} + (48.6 + 8)\dot{E} + 64E = 0 \quad (4\text{-}52)$$

It is apparent the second derivative signal will never cause saturation in a reverse direction, because the magnitude of feedback signal is even less than that in the semilinear case. But the velocity feedback signal is larger than that for the semilinear case, therefore, it will cause saturation in the opposite direction if this gain is large enough. In this example the second derivative is still needed to compensate the system but not as a control signal. The response of this system will be better than the original semilinear system, because its smallest real root is larger than that in the semilinear case. Since the system is operated in a continuous overdamped condition in the linear zone, then the trajectory follows the slow eigenplane in error space.

If three real roots are used in the compensated case, such as

$$r_1' = r_2' = r_3' = 4 \qquad \text{then } K_a = 0.0938,\ K_T = 0.625$$

the compensated characteristic equation is

$$\dddot{E} + (6 + 6)\ddot{E} + (40 + 8)\dot{E} + 64E = 0 \tag{4-53}$$

Equation of slow eigenplane is $\ddot{E} + 12\dot{E} + 16E = 0$ (4-54)

Equation of axisal plane is $\ddot{E} + \frac{K_T}{K_a}\dot{E} + \frac{1}{K_a}E = 0$ (4-55)

i.e. $E + 6.67E + 10.66E = 0$ (4-56)

By assuming $\ddot{E} = 0$ the slope of the slow eigenplane and the slope of the axisal plane, (the intersection with E vs $\dot{E}$ plane) are plotted in Fig. 4-4. Since the slope of the slow eigenplane is less negative than that of the axisal plane, there is no reverse saturation at any time. This step of the example is to confirm the statement that a critically damped system is preferable. Extending the definition of a semilinear system as that whenever the last part of the trajectory is linear the system is called a semilinear system, because in this example all the trajectories that start at the right side of the axisal plane will go to the slow eigenplane, and there is no overshoot or saturation in the opposite direction of the linear planary space. In other words, whenever the slow eigenspace is at the upper (less negative) side of the linear space or combined with the linear space, then the system is semilinear.

Returning to the discussion about saturated systems, usually the gain of the main channel of a saturated system is decided first. If the gain is very high, then the feedback signals must be large in order to make a critically or overdamped system. Since the slope of the linear space is inversely proportional to the magnitude of the feedback signals, but the slope of the eigenspace (critically damped) is increased with the gain, the system will naturally become a saturated system.

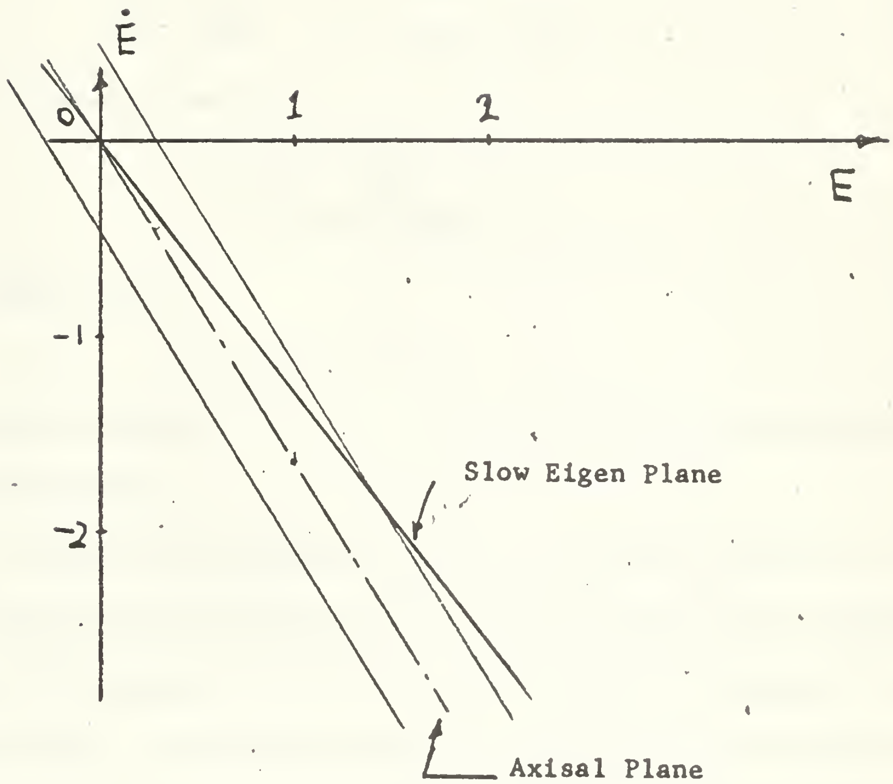


Fig. 4-4 Illustration of the Relative Position of Slow Eigen Plane and Axisal Plane.

In this third order example, if the gain is increased to $K = 512$, for the critically damped case $r_1' = r_2' = r_3' = 8$, the compensated characteristic equation is

$$\dddot{E} + (18 + 6)\ddot{E} + (184 + 8)\dot{E} + 512E = 0 \tag{4-57}$$

$$KK_a = 18, \qquad K_a = 0.035$$

$$KK_T = 184, \qquad K_T = 0.359$$

Equation of the axisal plane is

$$\ddot{E} + \frac{K_T}{K_a}\dot{E} + \frac{1}{K_a}E = 0 \tag{4-58}$$

i.e. $$\ddot{E} + 10.22\dot{E} + 28.5E = 0$$

Equation of eigenplane is

$$\ddot{E} + 16\ddot{E} + 64E = 0 \tag{4-59}$$

In this case the acceleration and the velocity feedbacks both are needed for compensating and control.

There is a case, the compensated system is critically damped, with three equal real roots, and the equations of the axisal plane and the eigenplane are the same, called normal critical case. If the gain increases, then the feedback signals will make the main amplifier saturated in the opposite direction. If the coefficients of the uncompsnsated characteristic equation are larger than the normal critical case, then the main amplifier gain can still be increased without saturation.

The normal critical case is a special semilinear case. From equation (4-40) the normal critical case for the above example is:

$$\dddot{E} + (KK_a + A)\ddot{E} + (KK_T + B)\dot{E} + KE = 0 \tag{4-60}$$

i.e. $$\dddot{E} + (8 + 16)\ddot{E} + (128 + 64)\dot{E} + 512E = 0 \tag{4-61}$$

$$K_a = \frac{1}{64}, \qquad K_T = 0.25$$

The equation of axisal plane is

$$\ddot{E} + \frac{K_T}{K_a}\dot{E} + \frac{1}{K_a}E = \ddot{E} + 16\dot{E} + 64E = 0 \tag{4-62}$$

The euqation of eigenplane is

$$\ddot{E} + (r_1 + r_2)\dot{E} + r_1 r_2 E = \ddot{E} + 16\dot{E} + 64E = 0 \tag{4-63}$$

Therefore, in order to test a system to determine whether feedback signals have caused reverse saturation, a procedure is to set up the normal semilinear equations and compare the corresponding coefficients in the uncompensated characteristic equation.

If the system is not critically damped, for example in order to make sure the system has no overshoot for maximum magnitude of input, the feedback signals must be increased to turn the slope of the slow eigenplane or axisal plane upward (less negative), and there are three unequal real roots. The method for testing this kind of system is indicated in the first example in this section.

The statements above are true for nth order systems. Since the equations for the nth order semilinear system have been derived it is easy to extend the things determined for third order system to nth order system. In general, when some of the coefficients of the uncompensated characteristic equation are large, or can be made large (for example by inner loop compensation method), then it is possible to reduce the number of feedback or control signals.

SECTION VII - NUMERICAL EXAMPLES AND DESIGN PROCEDURE FOR SATURATED SYSTEMS

Third Order Example:

By taking the example used for the semilinear system, the original characteristic equation is $\dddot{E} + 6\ddot{E} + 8\dot{E} + 64E = 0$ (4-64)

Assume the maximum gain is 1000, the compensated system is critically damped, then $r_1 = r_2 = r_3 = 10$. The characteristic equation after feedback compensation is

$$\dddot{E} + (1000K_a + 6)\ddot{E} + (1000K_T + 8)\dot{E} + 1000E = 0 \quad (4\text{-}65)$$

$$\dddot{E} + 30\ddot{E} + 300\dot{E} + 1000E = 0 \quad (4\text{-}66)$$

Where $K_a = 0.024$, $K_T = 0.292$

The equation of the axisal plane is $K_a\ddot{E} + K_T\dot{E} + E = 0$ (4-67)

i.e. $0.024\ddot{E} + 0.292\dot{E} + E = 0$ (4-68)

When $\ddot{E} = 0$, $\dot{E} = -3.43E$ (4-69)

The equation of the slow eigenspace is

$$\ddot{E} + (r_1 + r_2)\dot{E} + r_1r_2E = 0 \quad (4\text{-}70a)$$

i.e. $$\ddot{E} + 20\dot{E} + 100E = 0 \quad (4\text{-}70b)$$

When $\ddot{E} = 0$, $\dot{E} = -5E$

Therefore the slope of the slow eigenplane is more negative than that of the axisal plane. The trajectory of the system will be saturated in the opposite direction when the main amplifier output is saturated due to the step input signal.

Fifth Order Example:

Again by taking the fifth order example used for the semilinear system: Let $r_1 = 2.0$, $r_2 = 3$, $r_3 = 4$, $r_4 = 5$, $r_5 = 7$, (for the purpose of making each root in the saturated system larger than the corresponding root in the semilinear system.) Then the system gain will be $K = 840$.

If the gain is not proper, then change the value of the roots proportionally.

The characteristic equation of this compensated system is

$$E^{(5)} + (\gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5)\ddddot{E} + (\gamma_1\gamma_2+\gamma_1\gamma_3+\cdots\gamma_4\gamma_5)\dddot{E}$$

$$+ (\gamma_1\gamma_2\gamma_3+\cdots+\gamma_3\gamma_4\gamma_5)\ddot{E} + (\gamma_1\gamma_2\gamma_3\gamma_4+\cdots\gamma_2\gamma_3\gamma_4\gamma_5)\dot{E} + \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5 E = 0 \qquad (4\text{-}71)$$

i.e. $$E^{(5)} + 21\ddddot{E} + 169\dddot{E} + 651\ddot{E} + 1098\dot{E} + 840E = 0 \qquad (4\text{-}72)$$

The uncompensated characteristic equation is

$$E^{(5)} + 10.5\ddddot{E} + 39.5\dddot{E} + 63\ddot{E} + 36\dot{E} + 216E = 0 \qquad (4\text{-}73)$$

The coefficients of feedback signals are:

$$K_1 = 0.0125$$

$$K_2 = 0.154$$

$$K_a = 0.7$$

$$K_T = 1.266$$

The equation of linear space is:

$$K_1\ddddot{E} + K_2\dddot{E} + K_a\ddot{E} + K_T\dot{E} + E = 0 \qquad (4\text{-}74)$$

i.e. $$\ddddot{E} + 12.5\dddot{E} + 54\ddot{E} + 105\dot{E} + 80E = 0 \qquad (4\text{-}75)$$

The equation of slow eigenspace is

$$\ddddot{E} + (\gamma_1+\gamma_2+\gamma_3+\gamma_4)\dddot{E} + (\gamma_1\gamma_2+\cdots+\gamma_3\gamma_4)\ddot{E}$$

$$+ (\gamma_1\gamma_2\gamma_3+\cdots+\gamma_2\gamma_3\gamma_4)\dot{E} + \gamma_1\gamma_2\gamma_3\gamma_4 E = 0 \qquad (4\text{-}76)$$

i.e. $$\ddddot{E} + 14\dddot{E} + 71\ddot{E} + 124\dot{E} + 120E = 0 \qquad (4\text{-}77)$$

By comparing the coefficients of the corresponding terms in Eq. (4-75) and (4-77) the slope of the slow eigenspace is more negative than the slope of the linear space. Then the trajectory will cross the linear space and saturate in the opposite direction provided that the gain is high enough to cause saturation.

In the above two examples, the form of the die-out trajectory and the saturation level haven't been discussed. In order to design a saturated system without overshoot for maximum magnitude of the input signal the die-out trajectory must be found by calculation or by computer. This is not easy even for a Third order system. Then the analog computer setup is a preferable method for design. The effect of saturation level can also be found. The general design procedure using the analog computer can be arranged as follows:

(a) Set up the analog computer according to the uncompensated system equation but with adjustable feedback paths.

(b) Setting the main amplifier gain (saturation level) according to the specification.

(c) Using large amounts of feedback and maximum step input, record phase trajectories in every coordinate plane.

(d) Adjust feedback signals to make the maximum step input have optimum response in the E vs $\dot{E}$ plane.

(e) Reduce high order derivative feedback as much as possible but still obtain optimum response in E vs $\dot{E}$ plane.

(f) From the plots obtained in every coordinate plane find out the coordinates of the point in linear space through which the trajectory passes.

(g) Connect a line from the origin to this point found in (f). This is the line in linear space through which all the smaller step responses pass.

(h) Record the coefficients of the feedback signals and construct the linear space according to the equation in examples.

(i) Check the roots of the characteristic equation of the compensated system to see whether they are larger than those for the semilinear system.

NOTE:

In the actual physical system only the E vs $\dot{E}$ optimization curve is the important one. Then for problems involving step input the design parameters are not critical. In order to make the system have optimum response for various initial conditions, the selected linear space should be tested by setting initial conditions in the analog computer and making final adjustments. If additional damping is added to the system or the original die-out part of the trajectory is nearly a straight line in the E vs $\dot{E}$ plane, then the linear space selected nearly perpendicular to the E vs $\dot{E}$ plane is applicable. This approximation is based on the analysis of discontinuous damping in Chapter II.

SECTION VIII-COMPARISON AND DISCUSSION ABOUT SEMILINEAR AND SATURATED SYSTEMS

In a semilinear system the equation of slow eigenspace and linear space is decided by the characteristic equation of the uncompensated system. In other words the equations of slow eigenspace and linear space are combined into one and decided by the poles of the open loop transfer function. (See numerical examples). Then requiring a system with large value of poles to have no overshoot for any value of step inputs the semilinear system is the proper one to be used. But if the values of poles in the open loop transfer function are not large, then the combined linear and slow eigenspace will make the system have slow response. Therefore, a saturated system will be better. Because the linear space in a saturated system is decided by the optimum die-out trajectory of the maximum value of input signal, this linear space acts as a switching device to change the direction of the trajectory at a proper point in the error space. This corresponds to a high order bang-bang system, but without switching devices.

In a semilinear system the design procedure is quite simple and the result is the direct solution from the equations. In a saturated system, the die-out trajectory is decided by the motor-load combination and the optimization is only limited to a maximum value of step input. The linear space works as a straight line for switching to approximate the parabola die-out trajectory (the ideal switching line) in the second order case, or a plane to approximate the concaved plane in third order case. Such approximation is acceptable in some applications. On the other hand, the response can be improved by using non-linear feedback signals where the gain of the feedback signal is changing with the magnitude as indicated in Chapter II, or by using damping methods to make the die-out part of the trajectory stay nearly in a linear space as discussed in Chapter V and VI.

SECTION IX - DISCUSSION ABOUT SYSTEMS HAVING COMPLEX POLES IN THE OPEN LOOP TRANSFER FUNCTION

If the system has complex poles in the open loop transfer function the saturated trajectory will be more oscillatory. Actually this feedback compensation method does not require the original system to have good characteristics, the problem of feedback compensation is the gain limitation and saturation. But a semilinear system cannot be obtained in this case according to the relations given before.

When the system has complex poles in the open loop transfer function as in Fig. 4-5 , where $p_1 = -\alpha + j\omega, p_2 = -\alpha - j\omega$ then

$$G(S) = \frac{K_m}{S(S+\alpha+j\omega)(S+\alpha-j\omega)} = \frac{K_m}{S^3 + AS^2 + BS} \tag{4-78}$$

the characteristic equation without compensation feedback is

$$\dddot{E} + A\ddot{E} + B\dot{E} + KE = 0 \tag{4-79}$$

the compensated characteristic equation is

$$\dddot{E} + (KK_a + A)\ddot{E} + (KK_T + B)\dot{E} + KE = 0 \tag{4-80}$$

The equation of axisal plane is

$$K_a\ddot{E} + K_T\dot{E} + E = 0 \quad \text{or} \quad \ddot{E} + \frac{K_T}{K_a}\dot{E} + \frac{1}{K_a}E = 0 \tag{4-81}$$

and the equation of eigenplane is

$$\ddot{E} + (r_1 + r_2)\dot{E} + r_1 r_2 E = 0 \tag{4-82}$$

From Equation (4-81), (4-82)

$$K_a = \frac{1}{r_1 r_2}, \quad K_T = \frac{1}{K_a}(r_1 + r_2)$$

From Equation (4-3)

$$A = r_1 + r_2, \quad B = r_1 r_2 \tag{4-83}$$

then $r_1 = p_1$, $r_2 = p_2$. Since it has been assumed that r_1, r_2 & r_3 are real and p_1 and p_2 are complex, therefore there will be no eigenplane combined with the axisal plane unless the complex open loop poles are contained in the compensated characteristic equation.

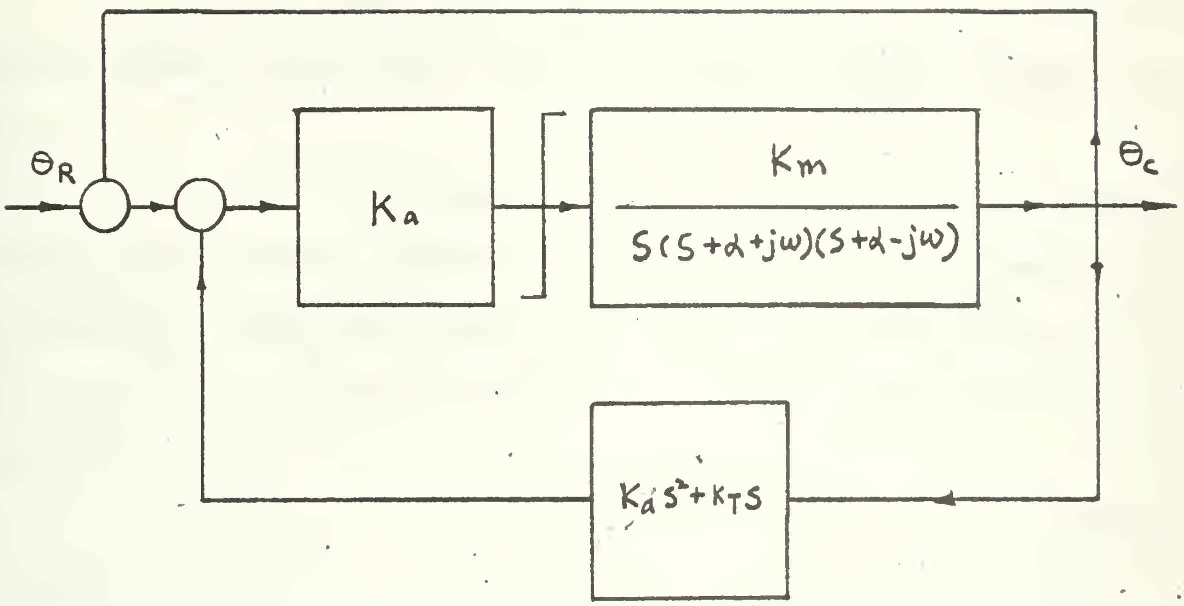


Fig. 4-5 Block Diagram of a Third Order System with Complex Open Loop Poles.

Example 1:

Let $P_1 = 1 + j2$. $P_2 = 1 - j2$, $K_m K_a = K = 27$

Then $$G(S) = \frac{27}{S^2 + 2S^2 + 5S} \tag{4-84}$$

and the compensated characteristic equation is

$$\dddot{E} + (KK_a + 2)\ddot{E} + (KK_T + 5)\dot{E} + 27E = 0 \tag{4-85}$$

The nearest plane in error space to this complex plane is when $r_1' = r_2' = \sqrt{5}$

that is

$$\ddot{E} + 4.46\dot{E} + 5E = 0 \tag{4-86}$$

Then if an approximate semilinear system is desired choose these two real roots in the compensated system. For $K = 27$ the three real roots will be $r_1 = r_2 = \sqrt{5}$, and $r_3 = 5.4$ and the characteristic equation is

$$\dddot{E} + 10.4\ddot{E} + 29\dot{E} + 27E = 0 \tag{4-87}$$

From Equation (4-85) $K_a = 0.311$

$K_T = 0.89$

The equation of the axisal plane is

$$\ddot{E} + 2.86\dot{E} + 3.21E = 0 \tag{4-88}$$

The equation of the slow eigenplane is

$$\ddot{E} + 4.46\dot{E} + 5E = 0 \tag{4-89}$$

In Fig. 4-6, E vs $\dot{E}$ and E vs $\ddot{E}$ planes indicate these two planes which were decided by (4-88), (4-89) and have an intersection on the E vs $\dot{E}$ plane.

In the positive $\ddot{E}$ half space and negative $\ddot{E}$ half space the relation between the slopes of slow eigenplane and axisal plane is changed, therefore the saturating amplifier will have reversed output.

Example 2:

Assume

$$G(s) = \frac{K}{S^3 + 4S^2 + 8S} \tag{4-90}$$

then the compensated characteristic equation is

$$\dddot{E} + (kk_a + 4)\ddot{E} + (kk_T + 8)\dot{E} + KE = 0 \tag{4-91}$$

select the slow eigenplane as

$$\ddot{E} + 4\dot{E} + 4E = 0 \qquad \text{where } r_1 = r_2 = 2 \tag{4-92}$$

for $K = 27$. The third root is $r_3 = 5.75$, then

$$K_a = 0.213$$
$$K_T = 0.704 \tag{4-93}$$

The equation of the axisal plane is

$$E + 3.3E + 1.42E = 0 \tag{4-94}$$

Here the equation of slow eigenplane is made to have the same coefficient as the second term of the uncompensated transfer function. The relative position between slow eigenplane and axisal plane of both cases are in Fig. 4-6.

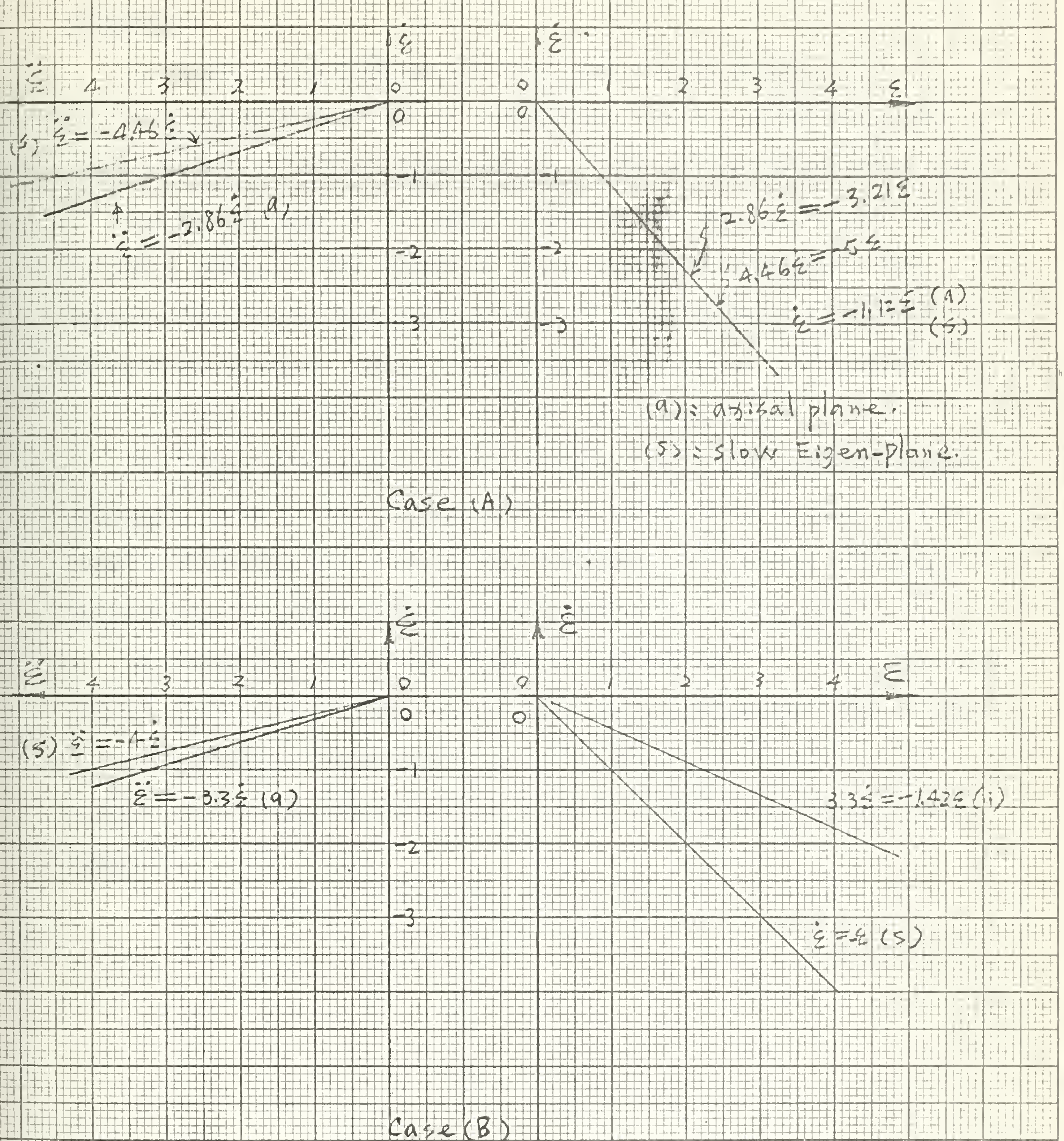


Fig. 4-6 Illustration of the Relative Positions of Slow Eigen Plane and Axisal Plane by the Intersections in the Coordinate Planes

Discussion:

a. About low gain system with open loop complex poles:

From the above analysis and examples the slow eigenplane and axisal plane cannot both have the same equation. This is true for any order system.

For some systems if the complex poles are not near the vertical axis in the s plane, a slow eigenplane can be made to stay largely in the linear planary space, especially when the gain is low and the thickness of the planary space is large. Then a semilinear system is obtained.

b. About high gain system with open loop complex poles:

For a high gain system, (especially when the original open loop poles are small) it is not suitable to use a semilinear system, because in that case the slow eigenplane must be very far apart from the fast eigenplane. The system will become slow in response even if the gain is high.

If a high gain "saturated" system is used the effect of these open loop complex poles are not large, because they can even be neglected if the term $K K_i \gg A_i$ where $K K_i$ is the feedback part and A_i is the original part of the coefficient of $\overset{i}{E}$ term in compensated characteristic equation.

SECTION X - SEMILINEAR AND SATURATED SYSTEM (General Case)

In the semilinear system the slow eigenspace has the same slope as that of the linear space. This is not the general case, also the fully saturated system is also an extreme case. In the usual case these two kinds of saturation may exist in one system and the trajectories can be divided into three categories as sketched in Fig. 4-7 for a third order case.

When the main amplifier gain can be increased without causing noise or disturbance, then it is best to move the slow eigenspace to have a more negative slope, i.e., to give a fast response.

The most desirable case is to use the critically damped case. As long as the coefficients of the original system characteristic equation (without damping) are less than the values in the normal critical case, then the slope of the critical eigenspace is always more negative than that of the linear space. Then a combined system will be formed naturally.

But if the maximum die-out trajectory tends to cause overshoot, then more feedback must be used to make the slope of the linear space less negative (that is to move the slow eigenspace and the linear space both to the less negative direction).

To make the system operate well for all initial conditions the same consideration upon the relative position of linear space and slow eigenspace must be defined as in a saturated system. Otherwise the system is only operating on a line or a curve in the whole space. This can never have good response for initial conditions other than the points on the calculated trajectory.

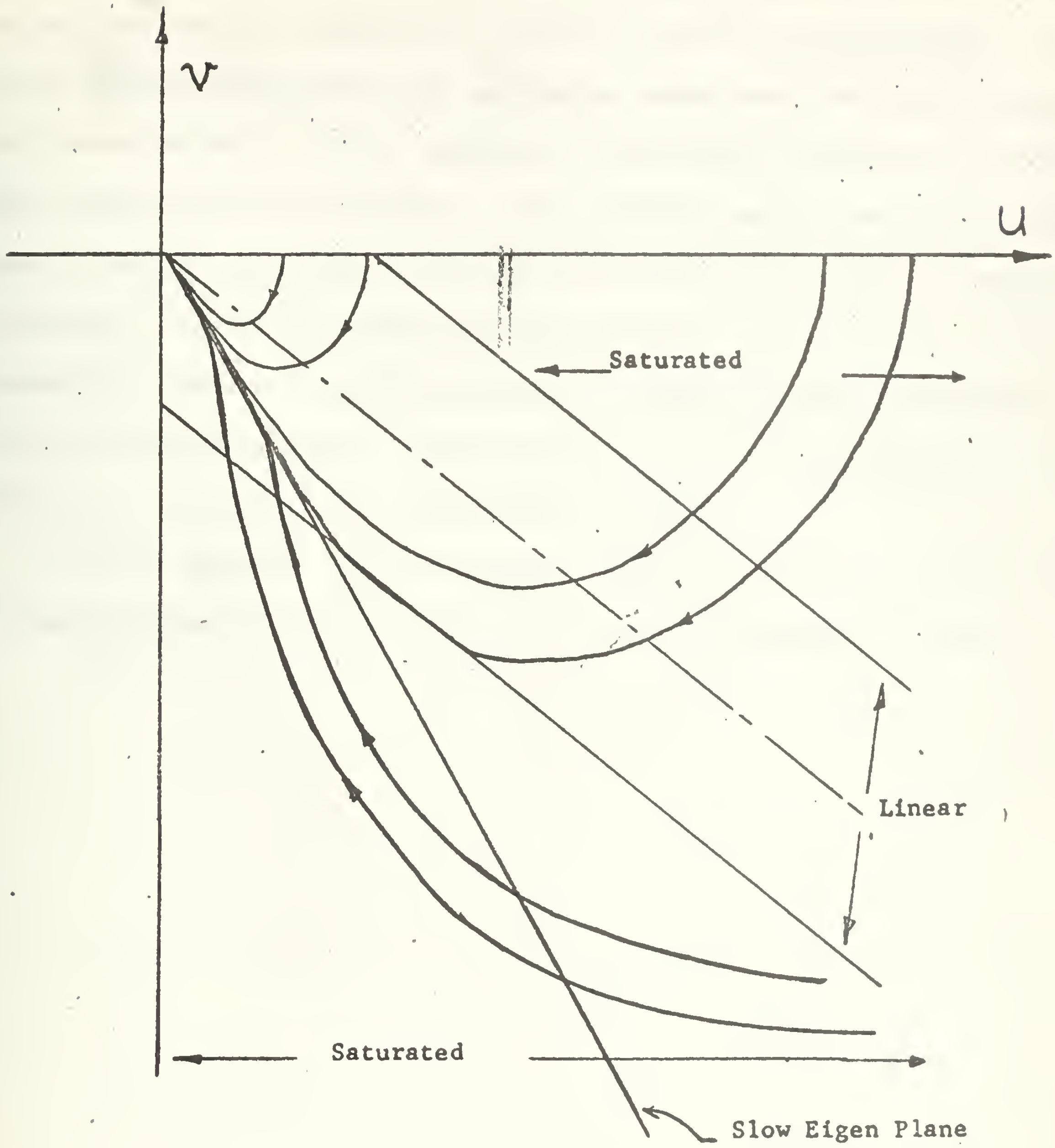


Fig. 4-7 The Projection of Trajectories onto a Plane Perpendicular to the Axisal Plane and Passing Through the Origin.

CONCLUSION

The addition of the phase space analysis method has shown a generalized method for moving the position of the linear space and eigen (vector) spaces which yield the solution to the continuous damped semilinear and saturated high order systems. For the semilinear system linear operation is obtained in the later part of the trajectory, for a saturated system, the linear space is used to switch the system to the opposite saturated direction. By proper adjustment a system with nearly optimum response can be obtained. This is especially true for the system with die-out trajectory having nearly the slope of the linear space. This means the closer the optimum die-out trajectory to the linear space, the better the result.

For the saturated system a nonlinear device can also be inserted in the feedback channel to make the linear space have a desirable curvature.

PART II - DETAIL CALCULATIONS AND COMPUTER RESULTS

SECTION I - COMPUTER STUDY OF A THIRD ORDER SEMILINEAR SYSTEM

From Fig. 4-8, Let $P_1 = 2.5 \quad P_2 = 2.7$, then the characteristic equation with compensation is

$$\dddot{E} + (K K_a + 5.25)\ddot{E} + (K K_T + 6.75)\dot{E} + KE = 0 \tag{4-95}$$

The equation of slow eigenplane and axisal plane is (from Eq. 4-4)

$$\ddot{E} + (2.5 + 2.7)\dot{E} + (2.5 \times 2.7)E = 0 \tag{4-96}$$

i.e. $$\ddot{E} + 5.2\dot{E} + 6.75E = 0 \tag{4-97}$$

For $K = 27$ then $\quad r_3 = 4, \quad K_a = \frac{1}{6.75} = 0.148, \quad K_T = 0.77$

For $K = 100$ then $\quad r_3 = 14.8, \quad K_a = 0.148, \quad K_T = 0.77$

This proves that the gain of the main amplifier does not effect the location of the slow eigenplane for this kind of system.

Equations for computer setup:

$$\begin{aligned}[-\bar{V}_S] &= K\left\{\frac{\alpha\theta_R}{\alpha V_S}\bar{\theta}_R - \frac{\alpha\theta_c}{\alpha V_S}\bar{\theta}_c - K_a\frac{\alpha\ddot{\theta}_c}{\alpha V_S}\bar{\ddot{\theta}}_c - K_T\frac{\alpha\dot{\theta}_c}{\alpha V_S}\bar{\dot{\theta}}_c\right\} \\ &= W_1\bar{\theta}_R - W_2\bar{\theta}_c - W_3\bar{\ddot{\theta}}_c - W_4\bar{\dot{\theta}}_c\end{aligned} \tag{4-98}$$

$$[\bar{J}] = \frac{\frac{\alpha V_S}{\alpha_J \alpha_t}[-\bar{V}_S]}{p + \frac{2.5}{\alpha_t}} = \frac{W_5}{p + W_6}[-\bar{V}_S] \tag{4-99}$$

$$[-\bar{\dot{\theta}}_c] = \frac{\frac{\alpha_J}{\alpha_{\dot{\theta}_c}\alpha_t}[\bar{J}]}{p + \frac{2.7}{\alpha_t}} = \frac{W_7}{p + W_8}[\bar{J}] \tag{4-100}$$

$$[\bar{\theta}_c] = \frac{1}{p}\left[\frac{\alpha\dot{\theta}_c}{\alpha_{\theta_c}\alpha_t}(-\bar{\dot{\theta}}_c)\right] = \frac{1}{p}W_9[-\bar{\dot{\theta}}_c] \tag{4-101}$$

$$[-\bar{\ddot{\theta}}_c] = p\left(\frac{\alpha\dot{\theta}_c}{\alpha\ddot{\theta}_c}\right)\bar{\dot{\theta}}_c = pW_{10}\bar{\dot{\theta}}_c \tag{4-102}$$

The circuit diagram of computer setup is in Fig. 4-9, and the numerical setting values for both K = 27 and K = 100 are in Table 4-1.

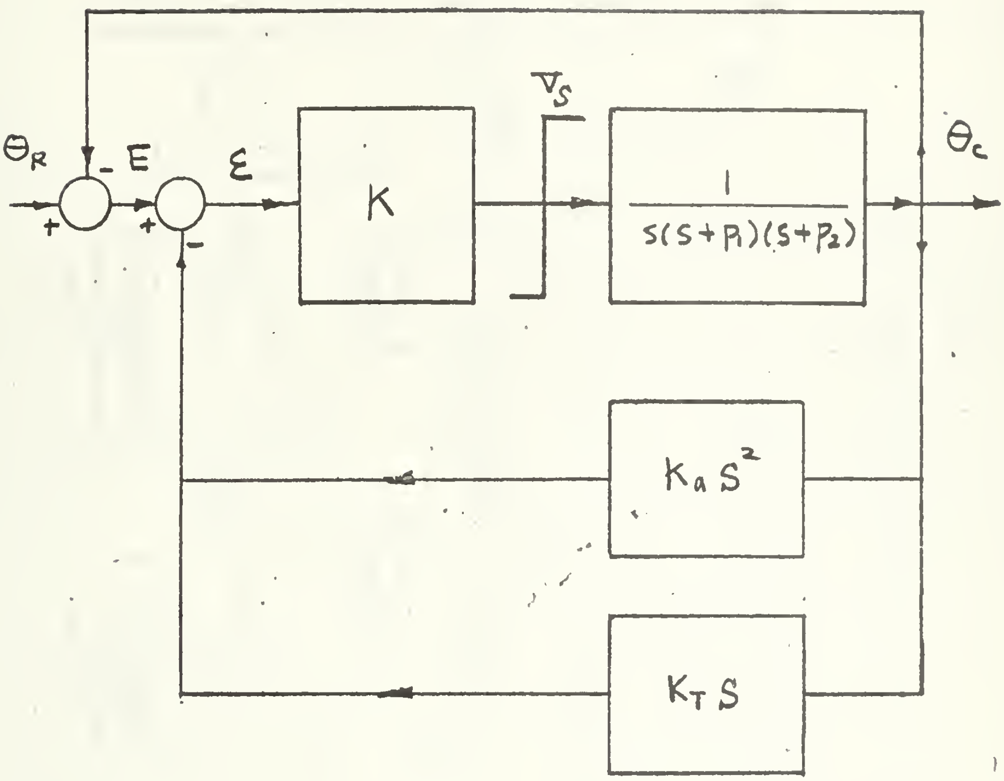


Fig. 4-8 Block Diagram of Third Order Feedback Control System with Saturated Main Amplifier.

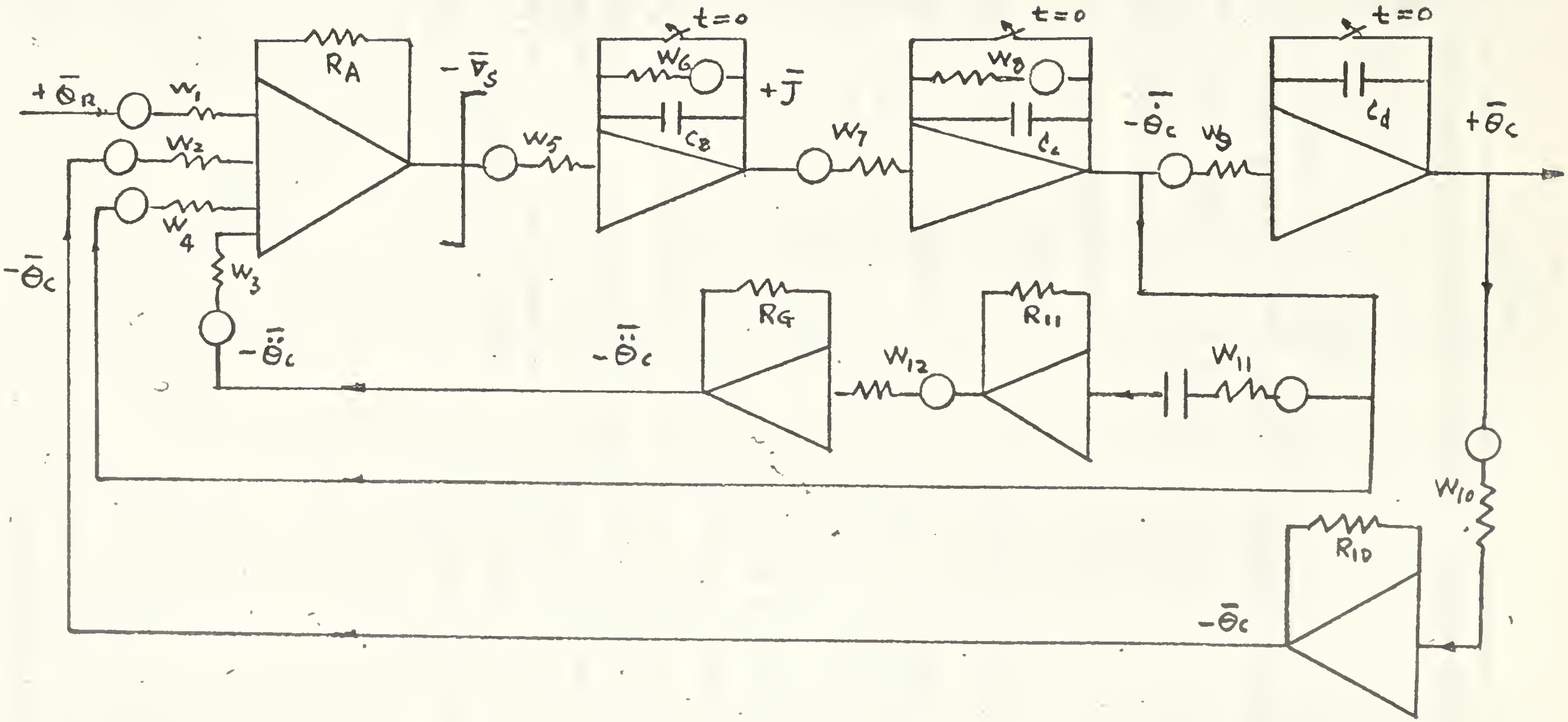


Fig. 4-9 Computer Setup for a Third Order Semilinear System.

TABLE 4-1

Data for computer setup for Third Order semilinear system.

$K = 27$ $\alpha_t = 1$

$\alpha_{\theta_R} = \alpha_{\theta_c} = 0.01$

$\alpha_{v_s} = 0.1$ $V_s = 10^v$ (Saturated main amplifier output)

$\alpha_J = 0.1$ $E_s = 3.7^v$

$\alpha_{\dot\theta_c} = 0.02$

$\alpha_{\ddot\theta_c} = 0.01$

						Dial Setting
W_1 =	2.7	$R_A = 10$	$R_1 = 1$	a_1=0.27	0.276	#1
W_2 =	2.7		$R_2 = 1$	a_2=0.27	0.276	#2
W_3 =	0.4		$R_3 = 1$	a_3=0.04	0.04	#3
W_4 =	4.15		$R_4 = 1$	a_4=0.415	0.4265	#4
W_5 =	1	C_B = 1.04	$R_5 = 0.5$	a_5=0.52	0.547	#5
W_6 =	2.5		$R_6 = 0.1$	a_6=0.26	0.317	#6
W_7 =	5	C_C = 1.04	$R_7 = 0.1$	a_7=0.52	0.641	#7
W_8 =	2.7		$R_8 = 0.1$	a_8=0.28	0.344	#8
W_9 =	2	C_d = 1.04	$R_9 = 0.1$	a_9=0.208	0.24	#9
W_{10}=	1	$R_E = 1$	R_{10}= 0.1	a_{10}= 0.5	pot	
W_{11}=	2	C_{11}=0.438	R_{11}= 10	a_{11}=0.456	0.568	#10
= a_{11} Rfgl or		C_{11} = 0.25	R_{11}= 10	a_{11} = a8	0.8825	#10
W_{12}=	1	$R_g = 1$	R_{12}= 0.5	a_{12}= 0.5	pot	

R in $M\Omega$

C in μf

K = 100				
W_1 =	10		R_1 = 0.2	a_1 = 0.2
W_2 =	10	R_A = 10	R_2 = 0.2	a_2 = 0.2
W_3 =	1.48		R_3 = 0.2	a_3 = 0.0296
W_4 =	15.4		R_4 = 0.2	a_4 = 0.308

From Fig. 4-10a and Fig. 4-10b the plots of the output voltages from the main amplifier indicate that there is no overshoot, this means that the main amplifier output voltage does not change its polarity, and full voltage is applied to the motor-load combination in the saturated region.

From Fig. 4-11 a space model can be made. The trajectories after entering into the linear planary space always tend to become tangent to the axisal plane. This cannot be shown in the E vs $\dot{E}$ plane, because the recorded curves in E vs $\dot{E}$ plane are the projections of the trajectories in space.

Fig. 4-12, illustrates the trajectories are not affected very much by using a R-C network to replace operational amplifier.

Fig. 4-13 is the frequency response curves. For small input there is no saturation at all, then the frequency response curve is that of a linear system. But when the magnitude of the input signal becomes larger and larger, then the saturation effect causes the frequency response curves to have more attenuation than the small signal case.

Discussion about the effect of saturating level and main amplifier gain

From the Brush recordings in Fig. 4-14 the narrower the linear planary space the more the overshoot of the main amplifier output voltage (due to the imperfect differentiation). Because for a very narrow linear zone the trajectory suddenly changes its direction at the edge of the linear zone, then the change of the corresponding high order derivatative is quite large. The differentiator cannot follow.

Using a small R-C product in the circuitor in the differentiator amplifier then the voltage drop will be large, and the noise problem becomes predominant.

When the linear (space) zone becomes large this situation is reduced proportionally. Computer results shown in Fig. 4-14.

The little ripple appears at the output of main amplifier does not cause oscillation in output of Θ_c or $\dot{\Theta}_c$, the effect is to cause a sharp edge in the $\ddot{\Theta}_c$ signal or higher derivative signal in higher order systems.

In the x-y plotter recording in Fig. 4-16 and Fig. 4-17, also indicates the effect of main amplifier gain and saturation level.

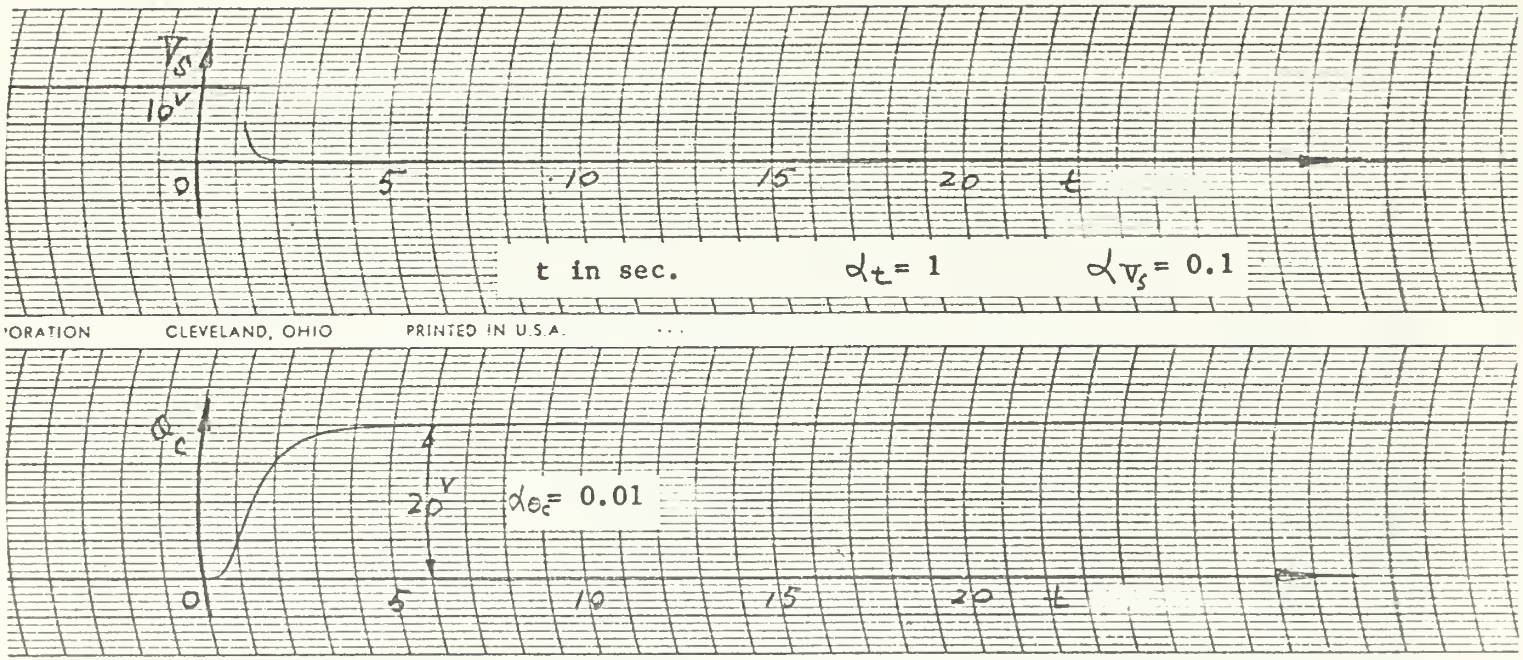


Fig. 4-10a Typical Step Response Curve for Medium Size of Input.
(with R-C generated)
(Computer Result $K=27$, $V_S=10^V$)

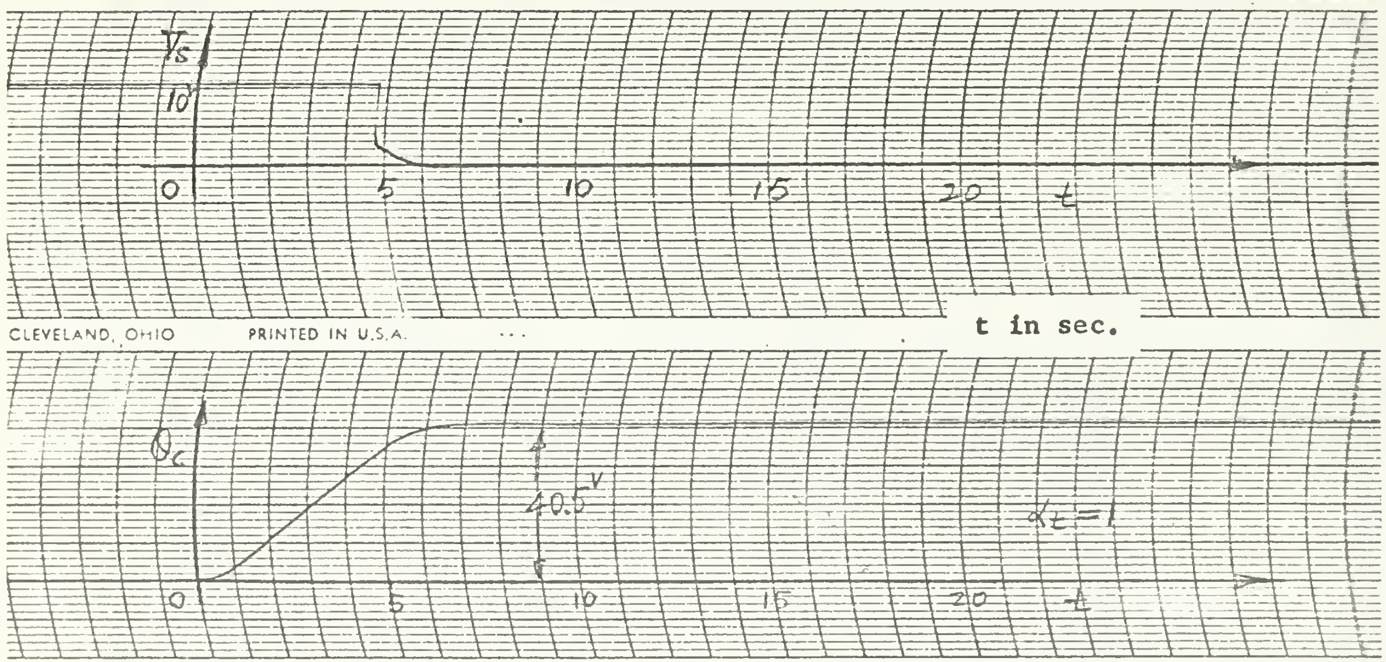


Fig. 4-10b Typical Response Curve for Large Size of Step Input
(Computer Result $K = 27$, $V_s = 10^v$)

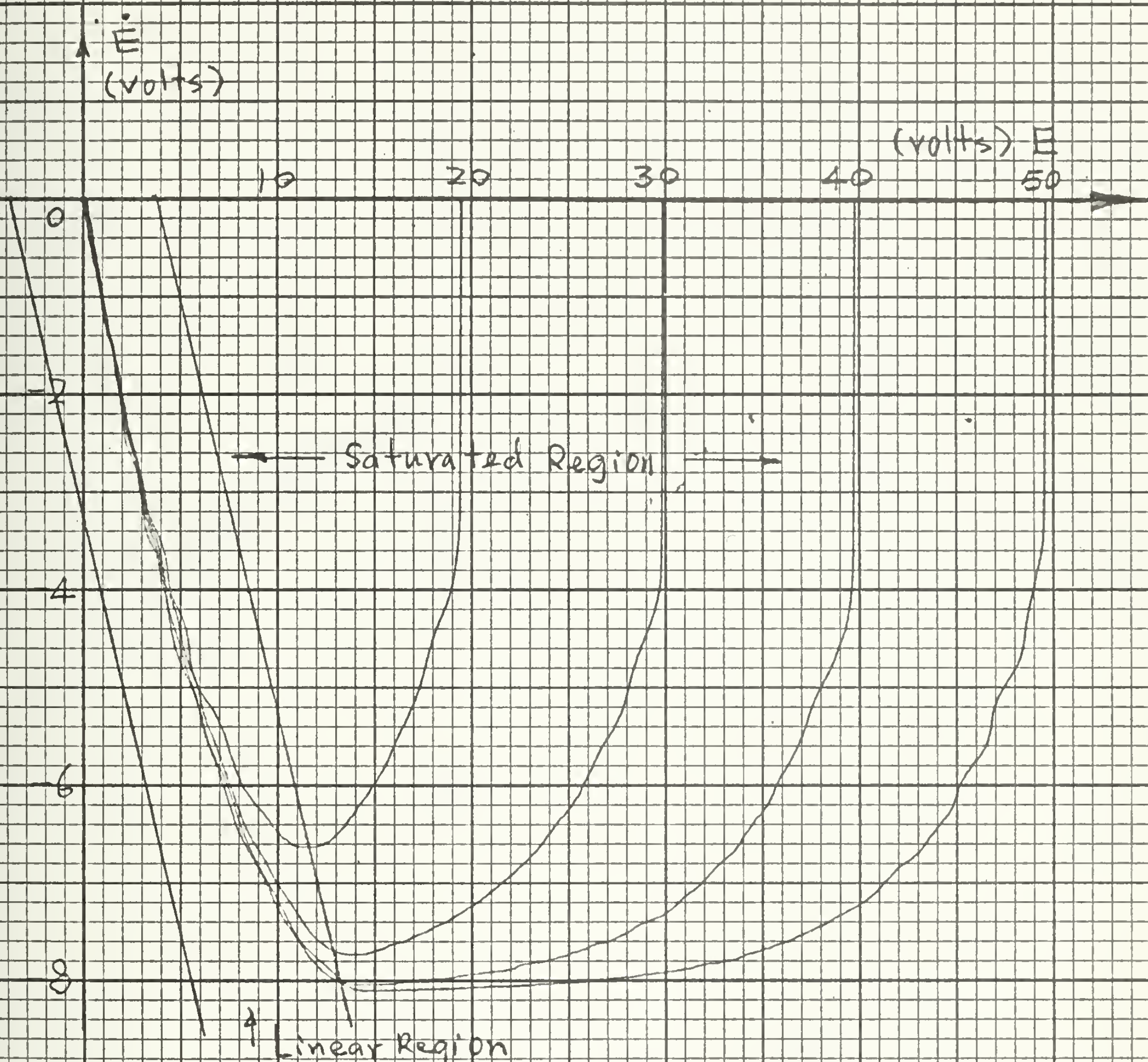


Fig. 4-11a E vs Ė plot of a Third Order Semilinear System with K = 27, $V_s = 10^v$ (Computer Results).

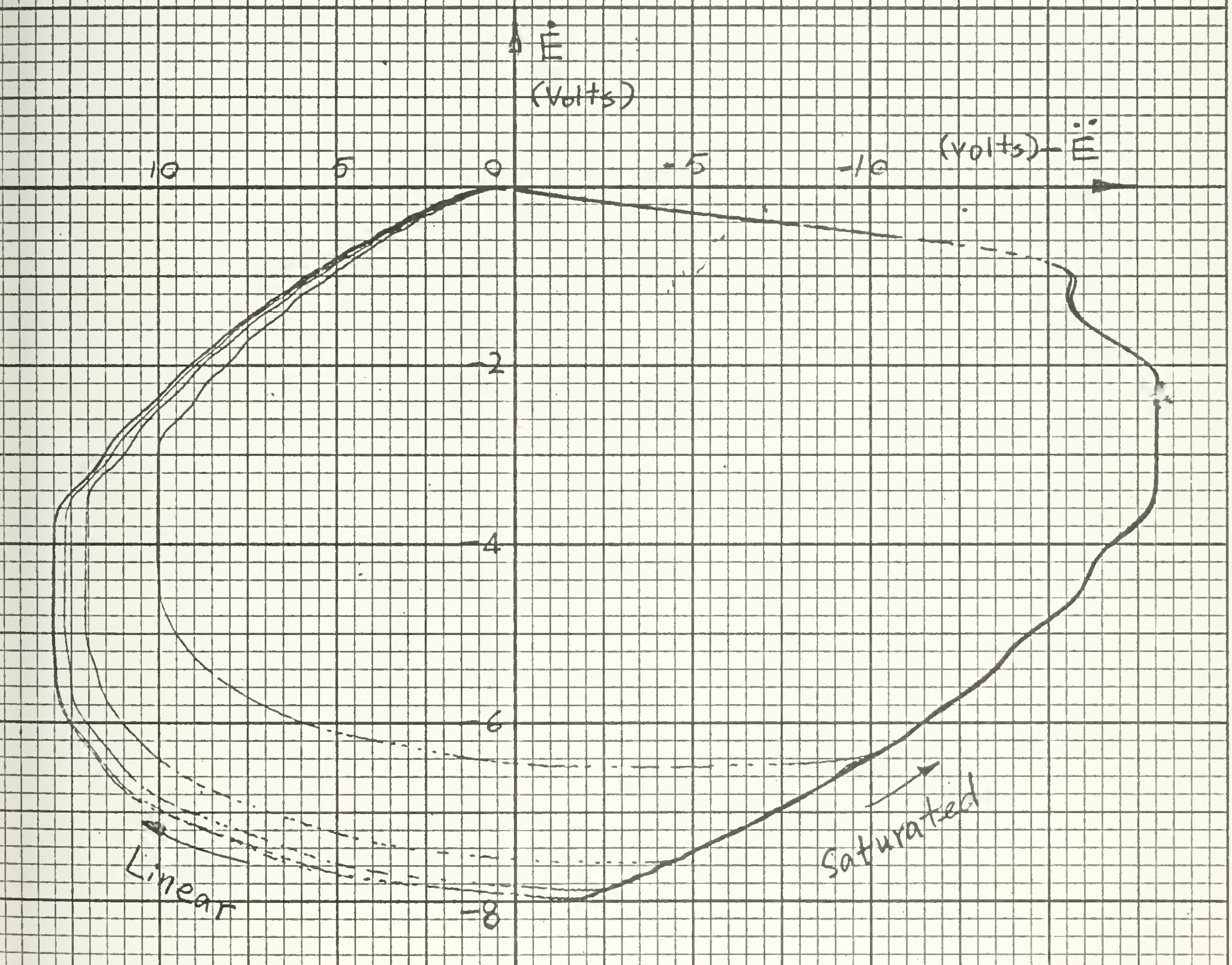


Fig. 4-11b $\ddot{E}$ vs $\dot{E}$ Plot

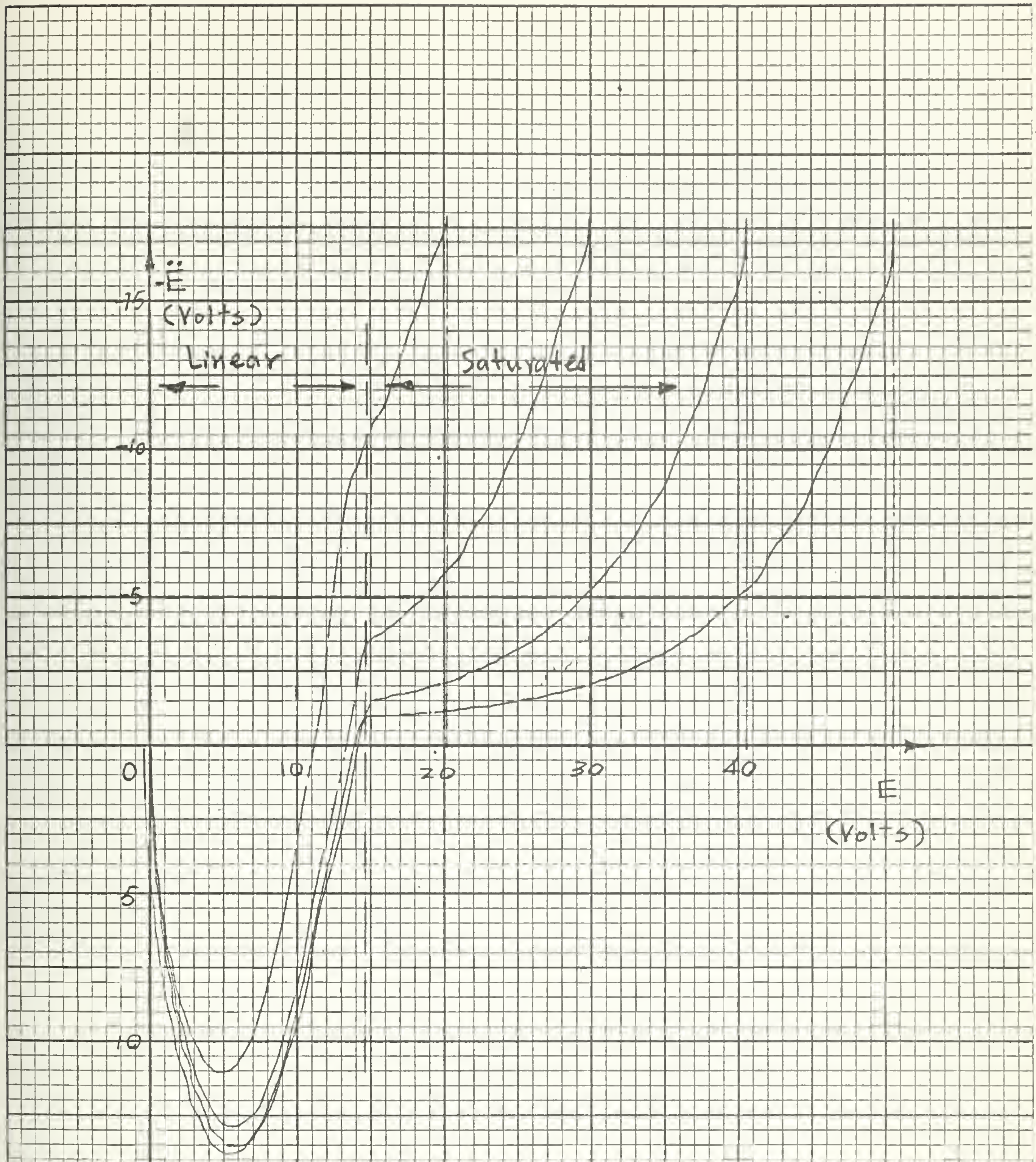


Fig. 4-11c E vs Ë Plot

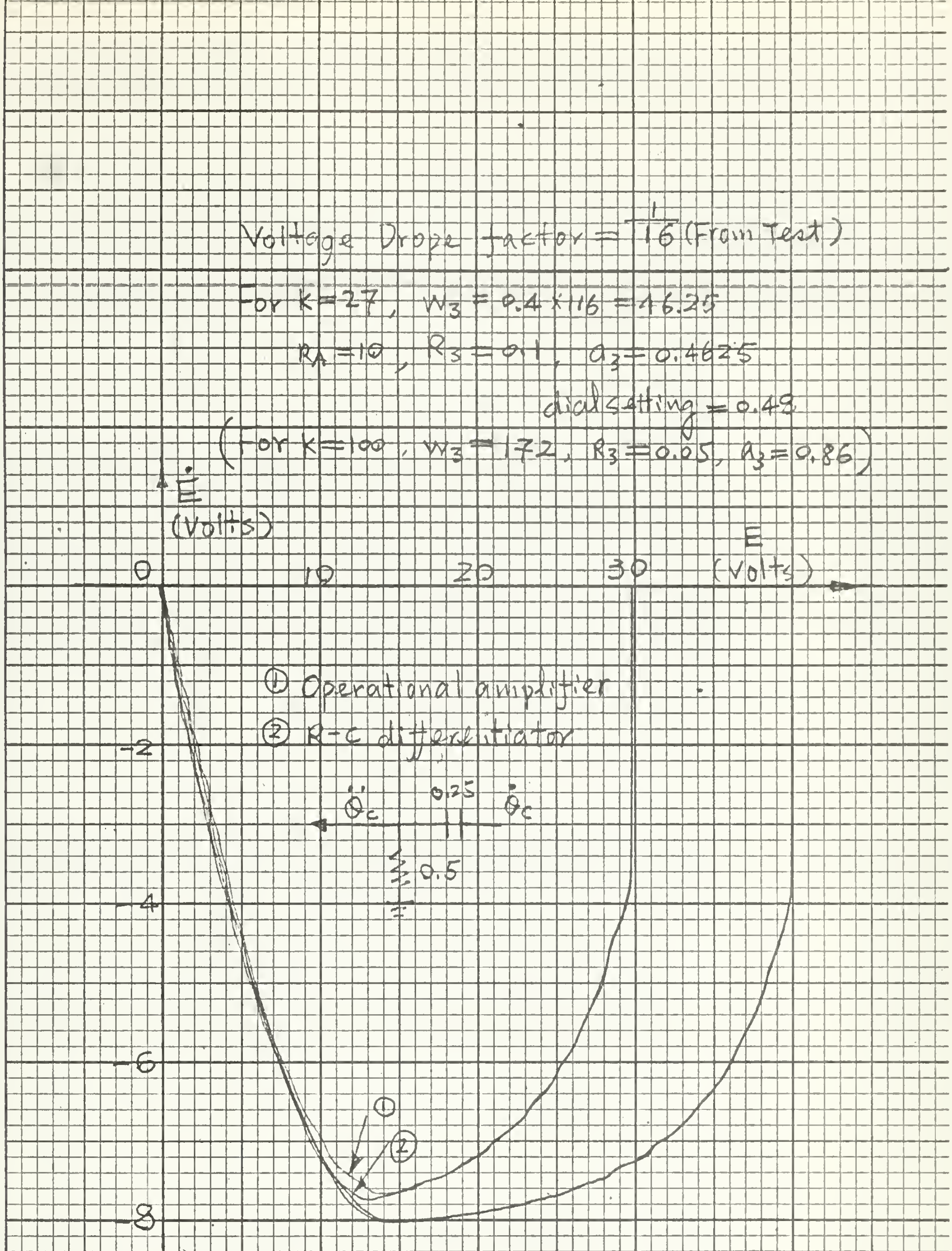


Fig. 4-12 E vs Ė Plot of Third Order Semilinear System with R-C Differentiator. K = 27.

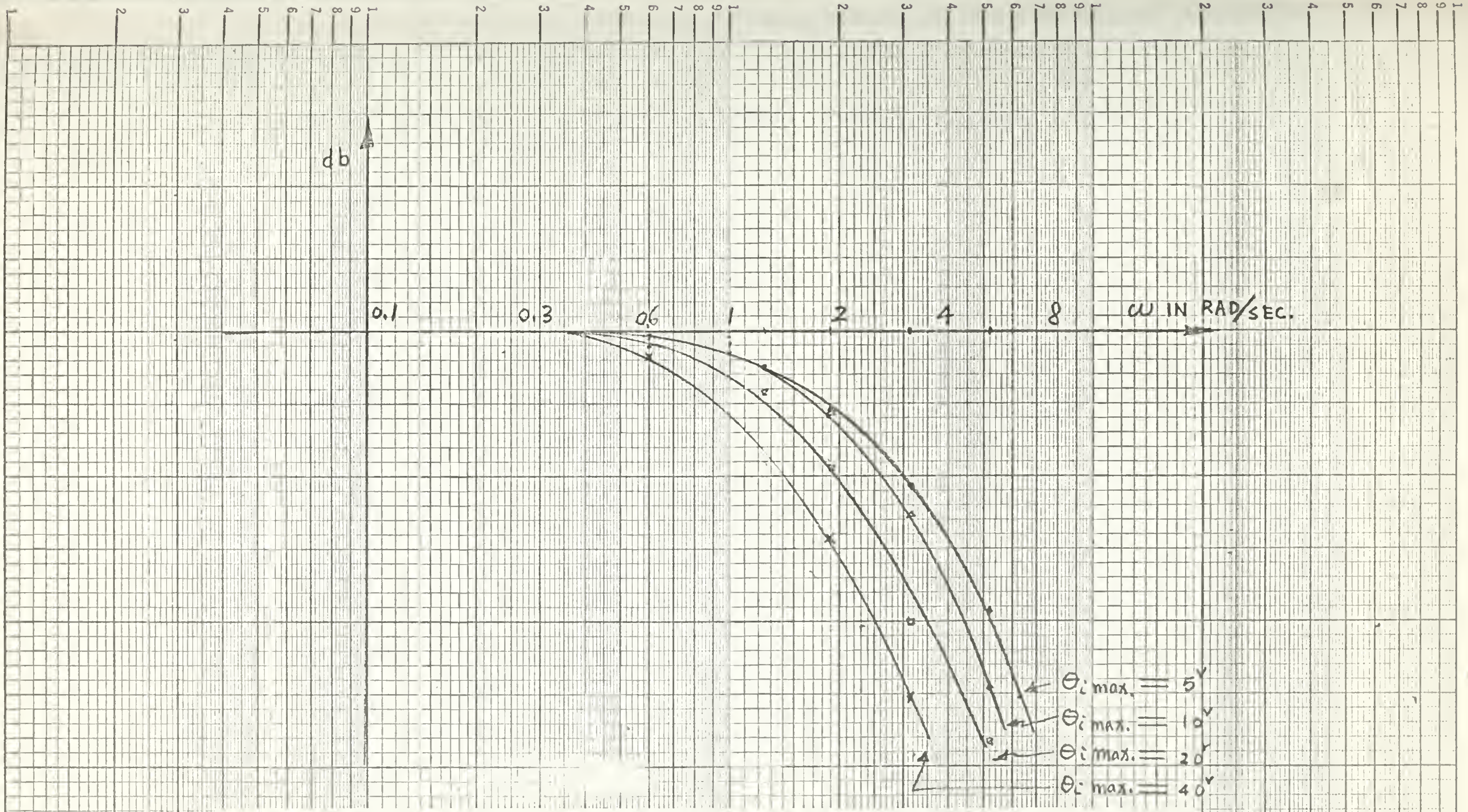


Fig. 4-13 Frequency Response Curves for a Third Order Semilinear System. (Computer Result K = 27, $V_S = 10^v$)

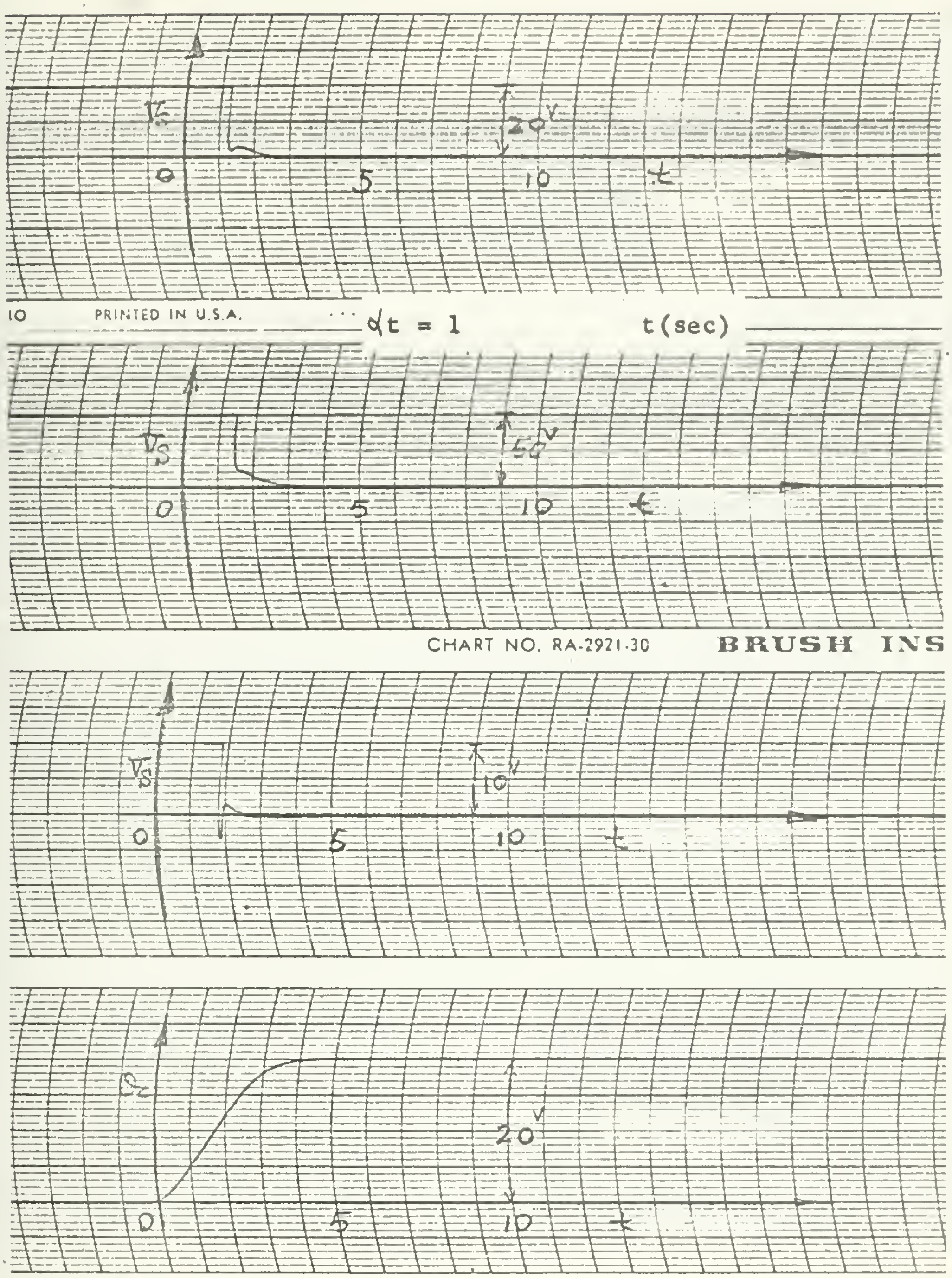


Fig. 4-14 Computer Recording for Illustrating the Effect of Saturating Level.

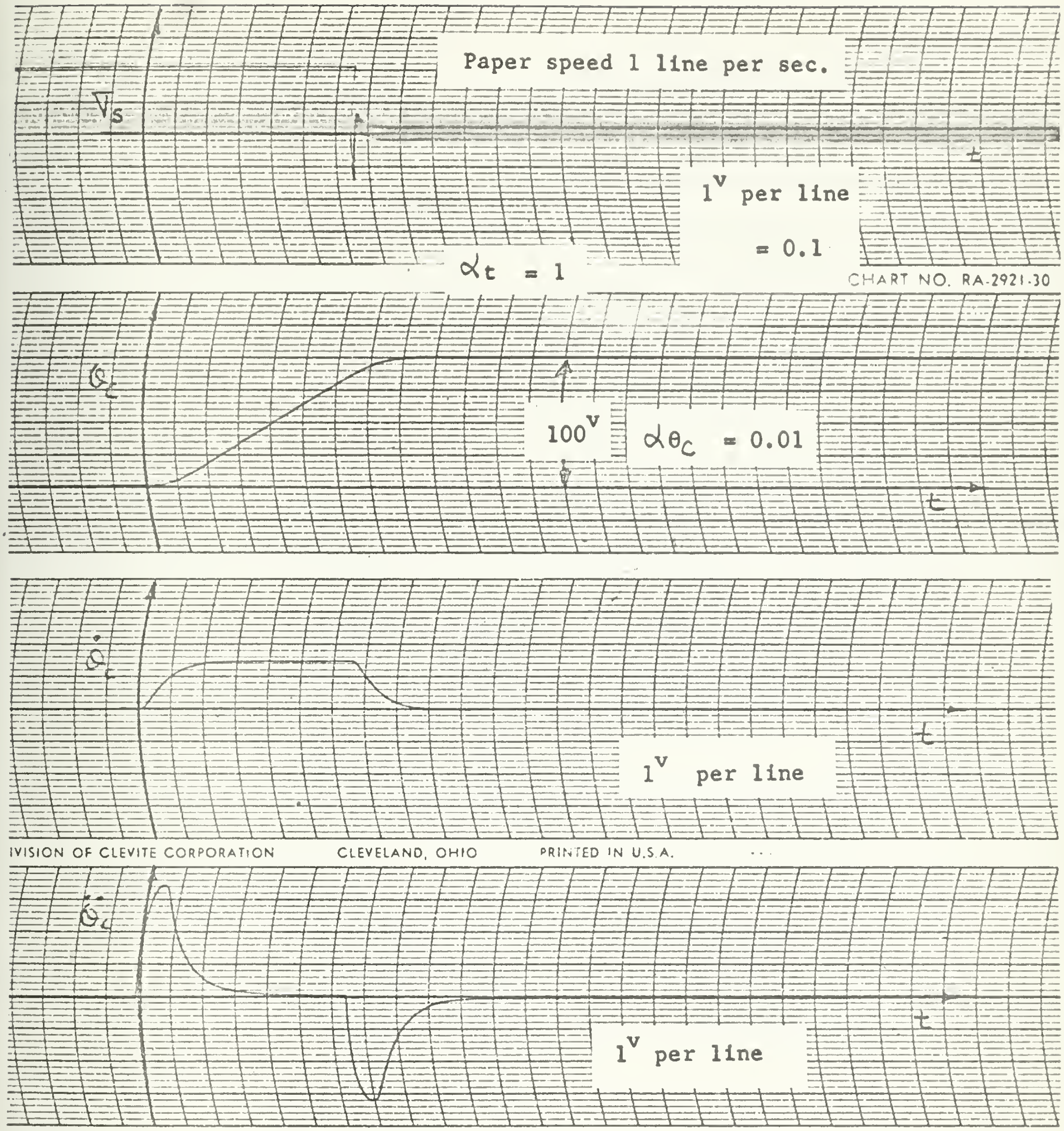


Fig. 4-15 Step Response Curves for Large Input.
(K = 100, $\bar{V}_s = 10^v$, Computer Result).

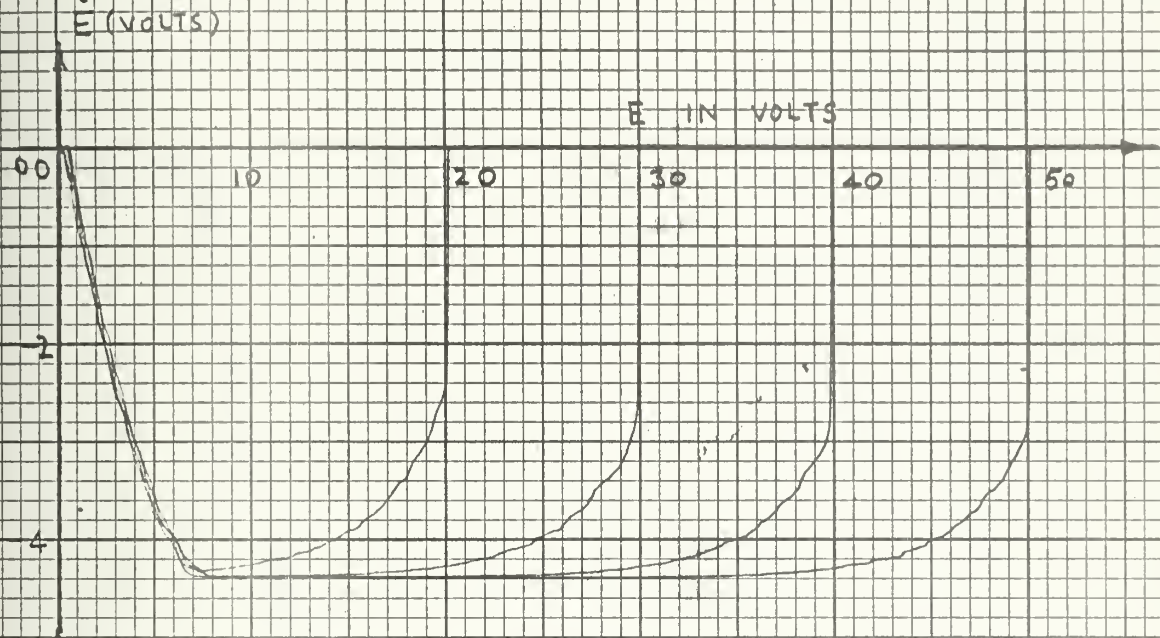


Fig. 4-16 Step Response of a Third Order Semilinear System with K = 100, $V_s = 5^V$.
(Computer Result)

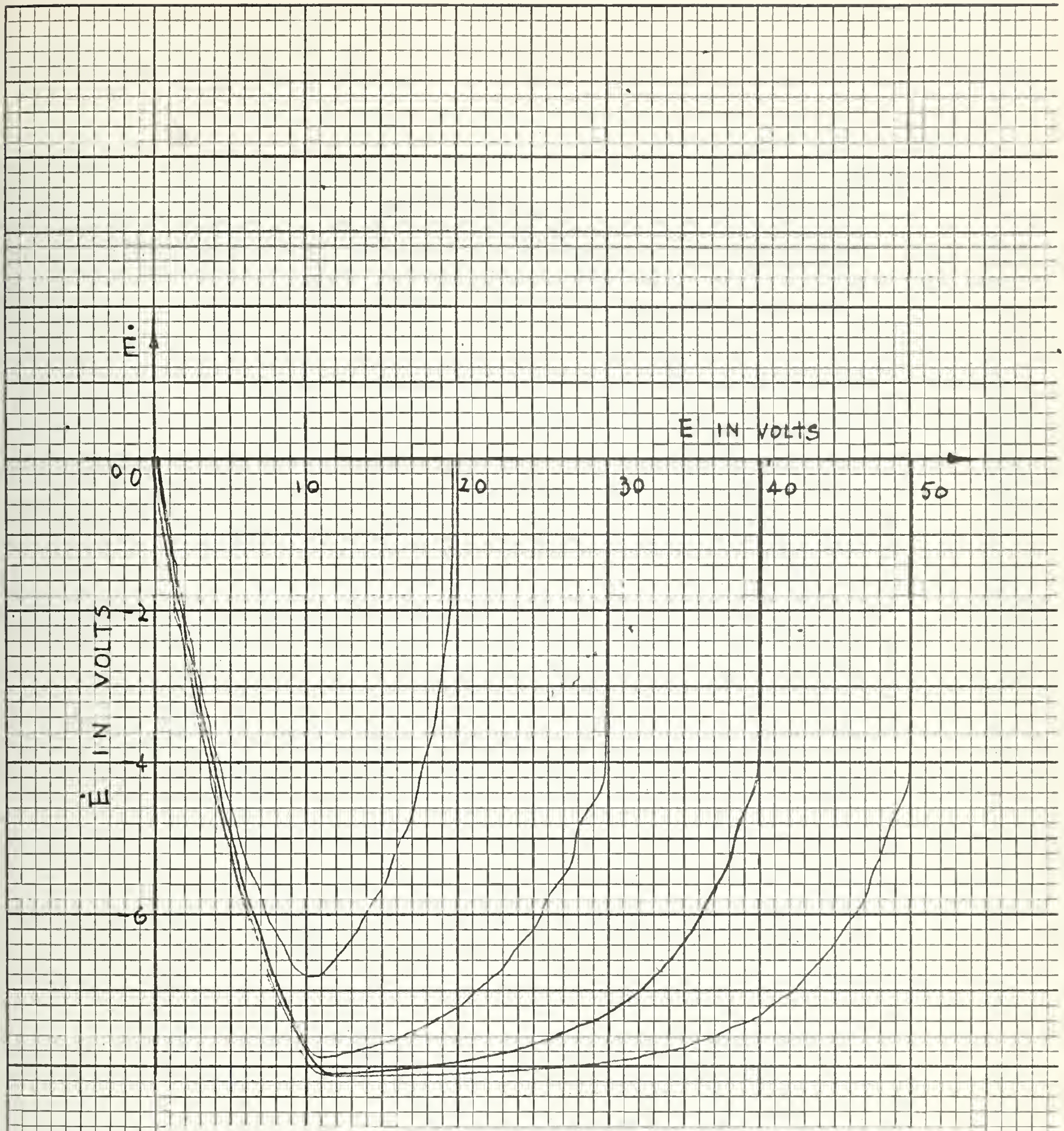


Fig. 4-17a Trajectories for a Third Order Semilinear System with K = 100 $V_S = 10^v$ (Computer Result)

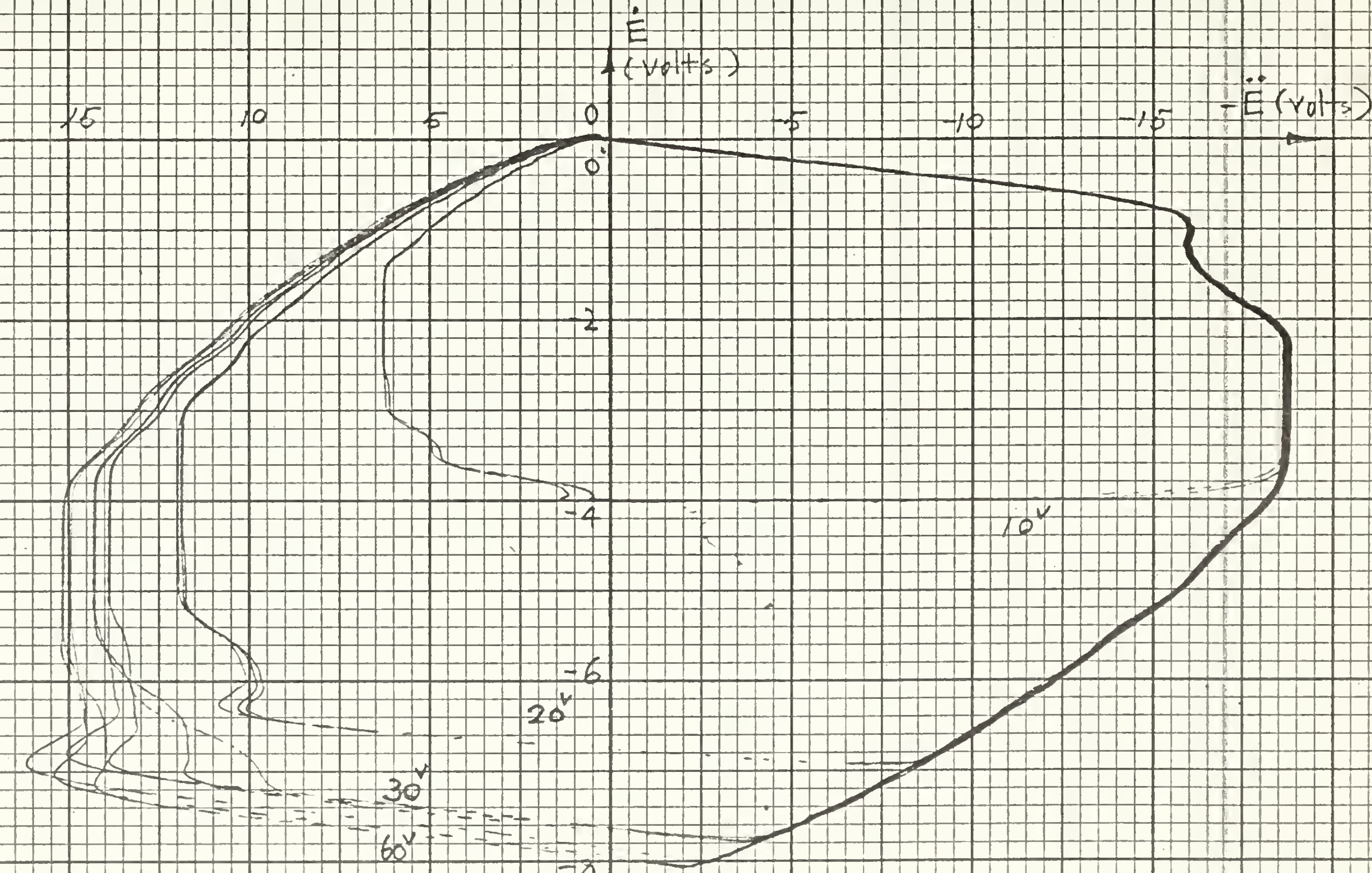


Fig. 4-17b E vs $\ddot{E}$ Plot for Different Magnitudes of Step Inputs.

SECTION II - COMPUTER STUDY FOR THIRD ORDER SEMILINEAR SYSTEM WITH SLOW EIGENPLANE WHICH HAS LARGER SLOPE THAN AXISAL PLANE.

From Section II of Part I of this Chapter, when the slow eigenspace has larger slope than the linear space, but the main amplifier gain is not high enough to cause the output voltage to be saturated in the reverse direction, then this is still a semilinear system. But the trajectory will have overshoot and saturate in the opposite direction if the input step signal is too large. The computer results for a third order system will verify this saturation.

Choosing the same open loop poles as in the above semilinear system

$$P_1 = 2.5, \quad P_2 = 2.7$$

Equation of compensated (continuously damped) system is

$$\dddot{E} + (KK_a + 5.25)\ddot{E} + (KK_T + 6.75)\dot{E} + KE = 0 \tag{4-103}$$

Equation of axisal plane is

$$K_a\ddot{E} + K_T\dot{E} + E = 0 \tag{4-104}$$

Equation of slow eigenplane is

$$\ddot{E} + (r_1 + r_2)\dot{E} + r_1 r_2 E = 0 \tag{4-105}$$

where $r_1 \leq r_2 \leq r_3$

For $K = 100$

by using critically damped case $r_1 = r_2 = r_3 = 4.64$

From Equation (4-103) $\dddot{E} + 13.92\ddot{E} + 64.8\dot{E} + 100E = 0$ (4-106)

Then

$$K_a = \frac{13.92 - 5.25}{100} = 0.0867$$

$$K_T = \frac{64.8 - 6.75}{100} = 0.5805$$

From Equation (4-104) $\ddot{E} + 6.71\dot{E} + 11.55E = 0$ (Axisal plane) (4-107)

From Equation (4-105) $\ddot{E} + 9.29\dot{E} + 21.5E = 0$ (4-108)

(slow or combined eigenplane)

Computer set up:

$$\left.\begin{aligned} W_1 = W_2 &= 10 \\ W_3 &= 0.867 \\ W_4 &= 11.61 \end{aligned}\right\} R_A = 10 \qquad \begin{aligned} R_1 = R_2 &= 0.2 \qquad a_1 = a_2 = 0.2 \\ R_3 &= 0.1 \qquad a_3 = 0.0867 \\ R_4 &= 0.2 \qquad a_4 = 0.233 \end{aligned}$$

$$W_{11} = 2 \qquad C_{11} = 0.25 \qquad R_{11} = 1 \qquad a_{11} = 0.8$$

Other setting values are same as before. The computer results are in Fig.4-18 and Fig.4-19.

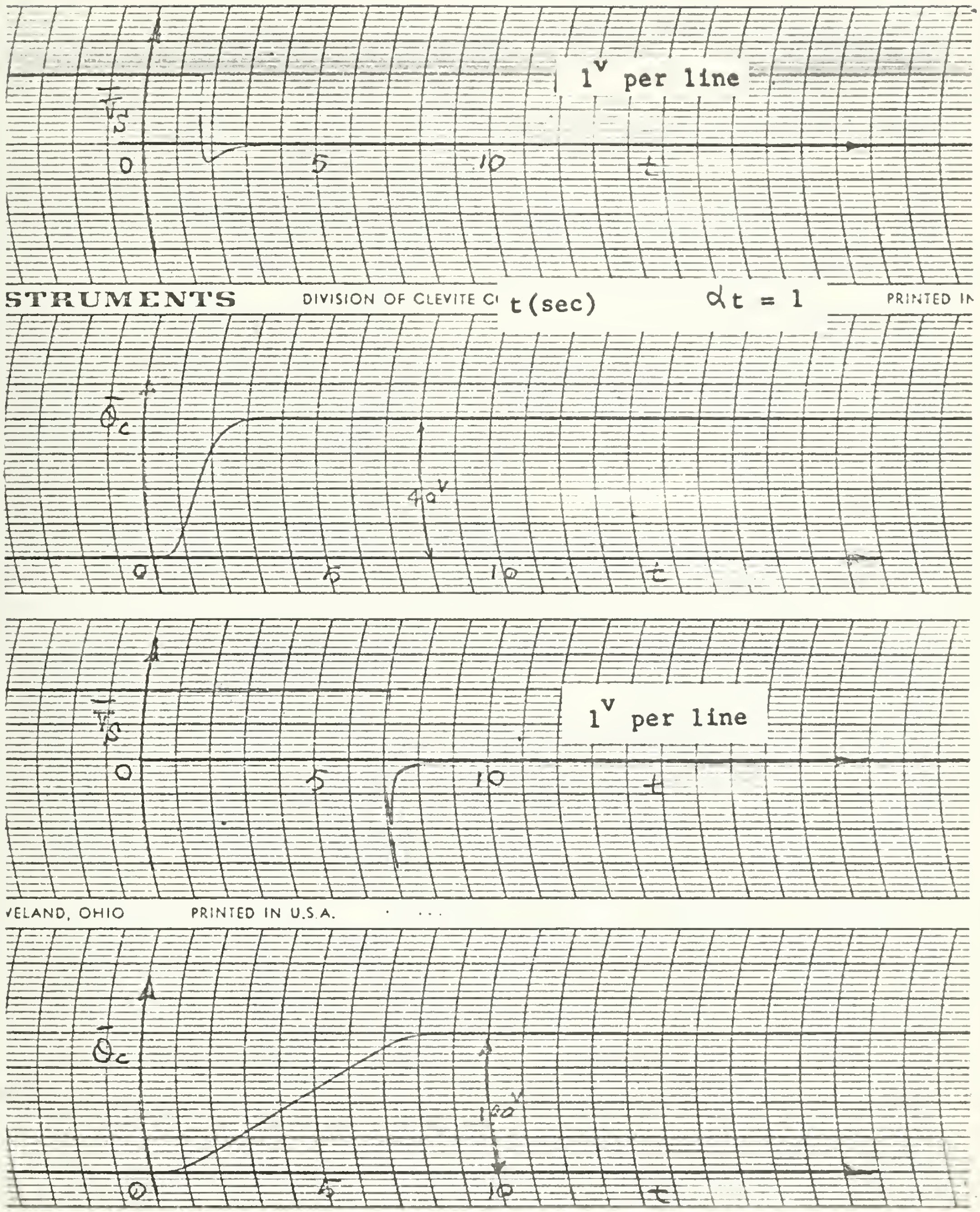


Fig. 4-18 An Illustration to Show the $\overline{V}_S$ Signal Changes Its Direction but not Saturated in Reversed Direction. (Computer Results, K = 100)

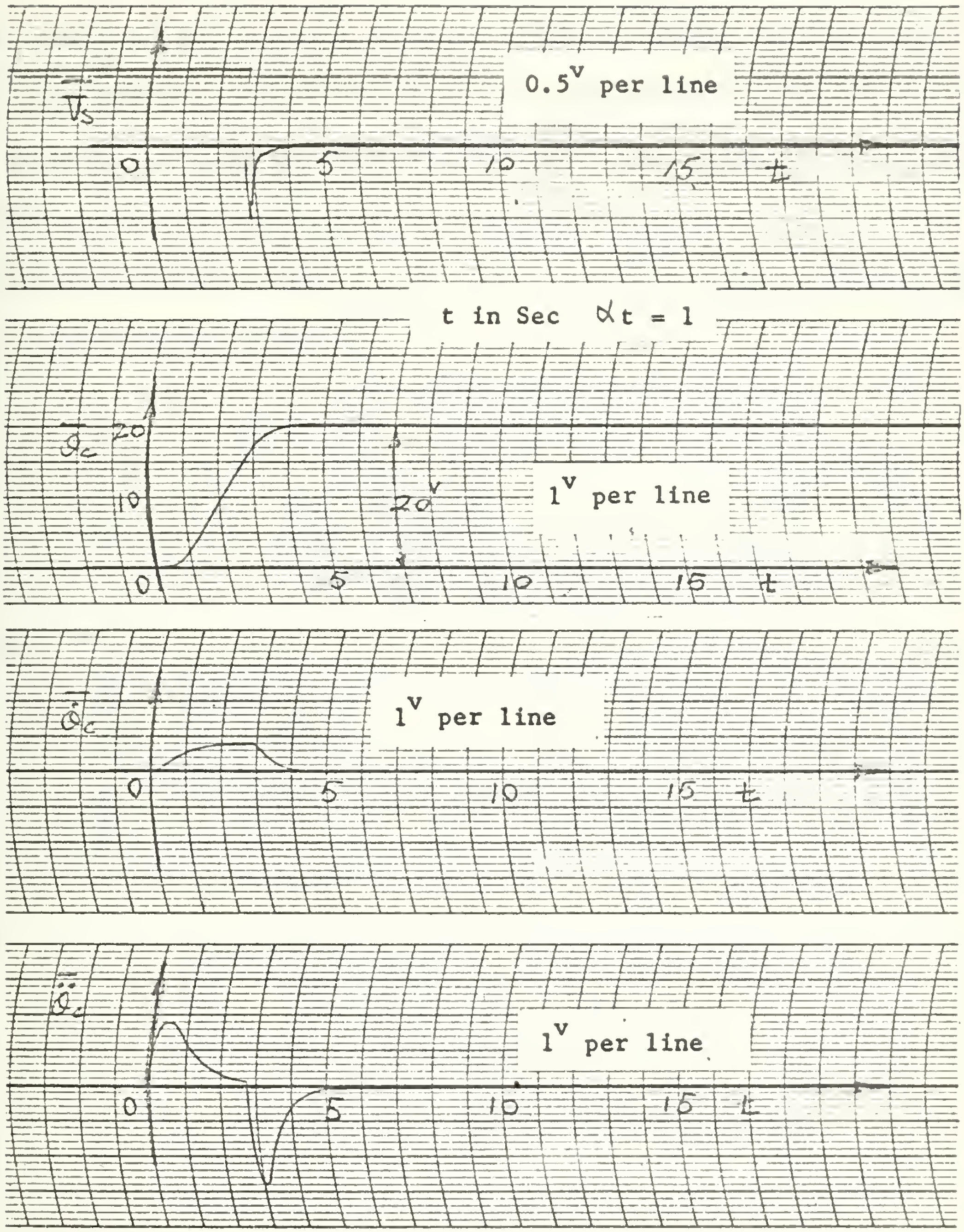


Fig. 4-19 Typical Response Curves of a Saturated Third Order System with Low Gain. (Computer Result, K = 100, $\bar{V}_S = 5.5^v$)

SECTION III - THIRD ORDER HIGH GAIN SATURATED SYSTEM.

For any critically damped system, if the gain is increased to some large value, the system will become a saturated system. Because in the characteristic equation (continuously feedback compensated) the feedback signal amplitude is increased, also the slope of the combined eigenvectors tend to be more negative than the axisal plane. Finally the original motor-load time constant even can be neglected.

In the previous system use $K = 255$, then for the critically damped case $r_1 = r_2 = r_3 = 6.35$

$$\dddot{E} + 19\ddot{E} + 121\dot{E} + 225 = 0 \tag{4-109}$$

$$K_a = 0.0539$$

$$K_T = 0.45$$

Computer setting

$$\left.\begin{aligned} W_1 = W_2 &= 25.5 \\ W_3 &= 1.37 \\ W_4 &= 23 \end{aligned}\right\} R_A = 10 \qquad \begin{aligned} R_1 = R_2 &= 0.2 \\ R_3 &= 0.1 \\ R_4 &= 0.2 \end{aligned} \qquad \begin{aligned} a_1 = a_2 &= 0.511 \\ a_3 &= 0.139 \\ a_4 &= 0.46 \end{aligned}$$

For $K = 511$, $r_1 = r_2 = r_3 = 8$

$$\dddot{E} + 24\ddot{E} + 192\dot{E} + 512E = 0 \tag{4-110}$$

$$K_a = 0.0366$$

$$K_T = 0.362$$

$$\left.\begin{aligned} W_1 = W_2 &= 51.1 \\ W_3 &= 1.88 \\ W_4 &= 37 \end{aligned}\right\} R_A = 10 \qquad \begin{aligned} R_1 = R_2 &= 0.1 \\ R_3 &= 0.1 \\ R_4 &= 0.1 \end{aligned} \qquad \begin{aligned} a_1 = a_2 &= 0.511 \\ a_3 &= 0.188 \\ a_4 &= 0.37 \end{aligned}$$

The results of computer solution are shown in Fig. 4-20 to Fig. 4-23. Fig. 4-20 shows the typical step response curves of saturated system with high main amplifier gain. Fig. 4-21 indicates the effect of saturation level. Fig. 4-22 is recorded to make a space model. That the linear planary space acts as a switching space can be clearly shown. And Fig. 4-23 shows the step response for very small and very large magnitude of input signal.

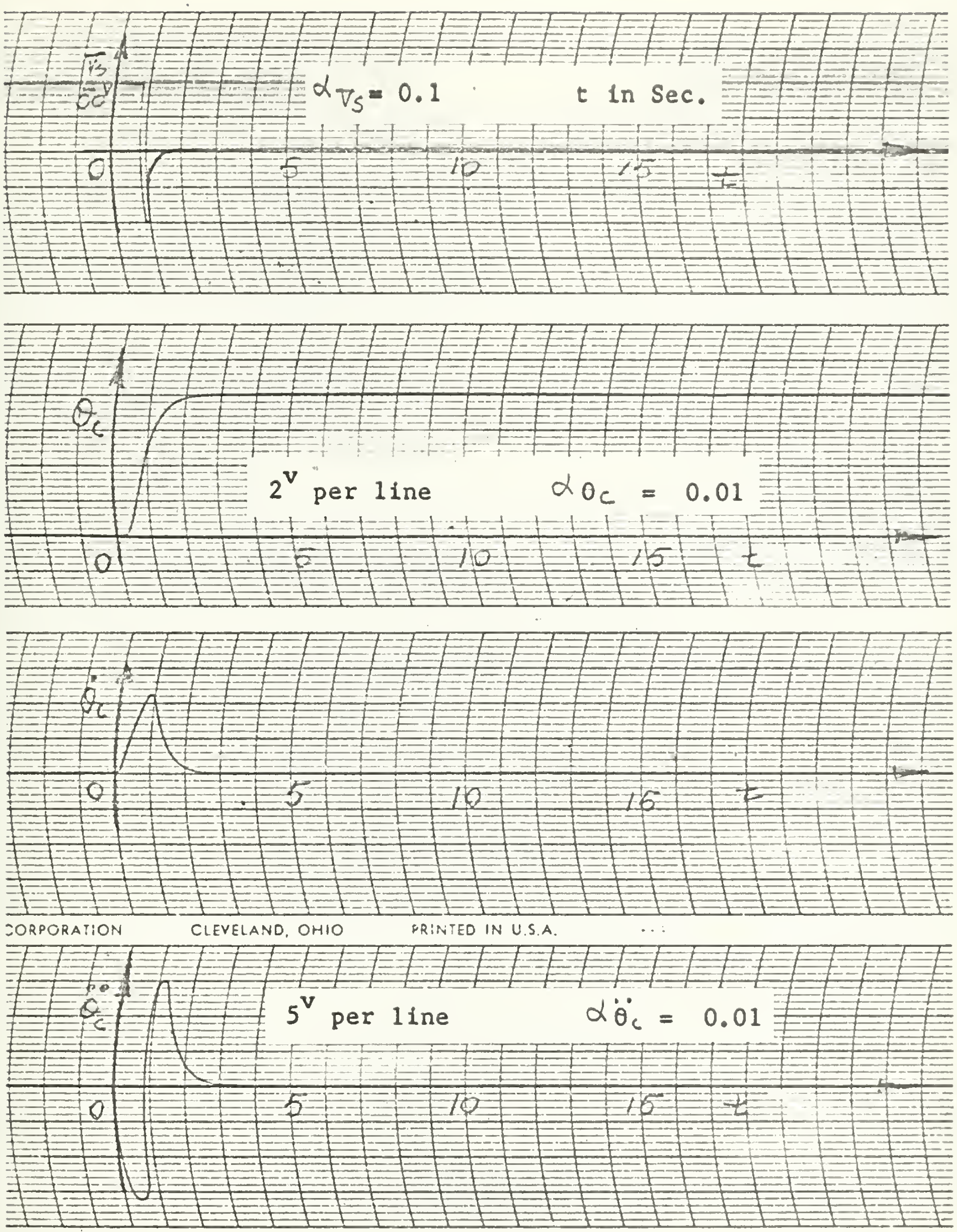


Fig. 4-20 Typical Response Curves of Saturated System (with Step Input) with High Gain. (Computer Results, K = 511, $\bar{V}_S = 50^V$) Critically Damped.

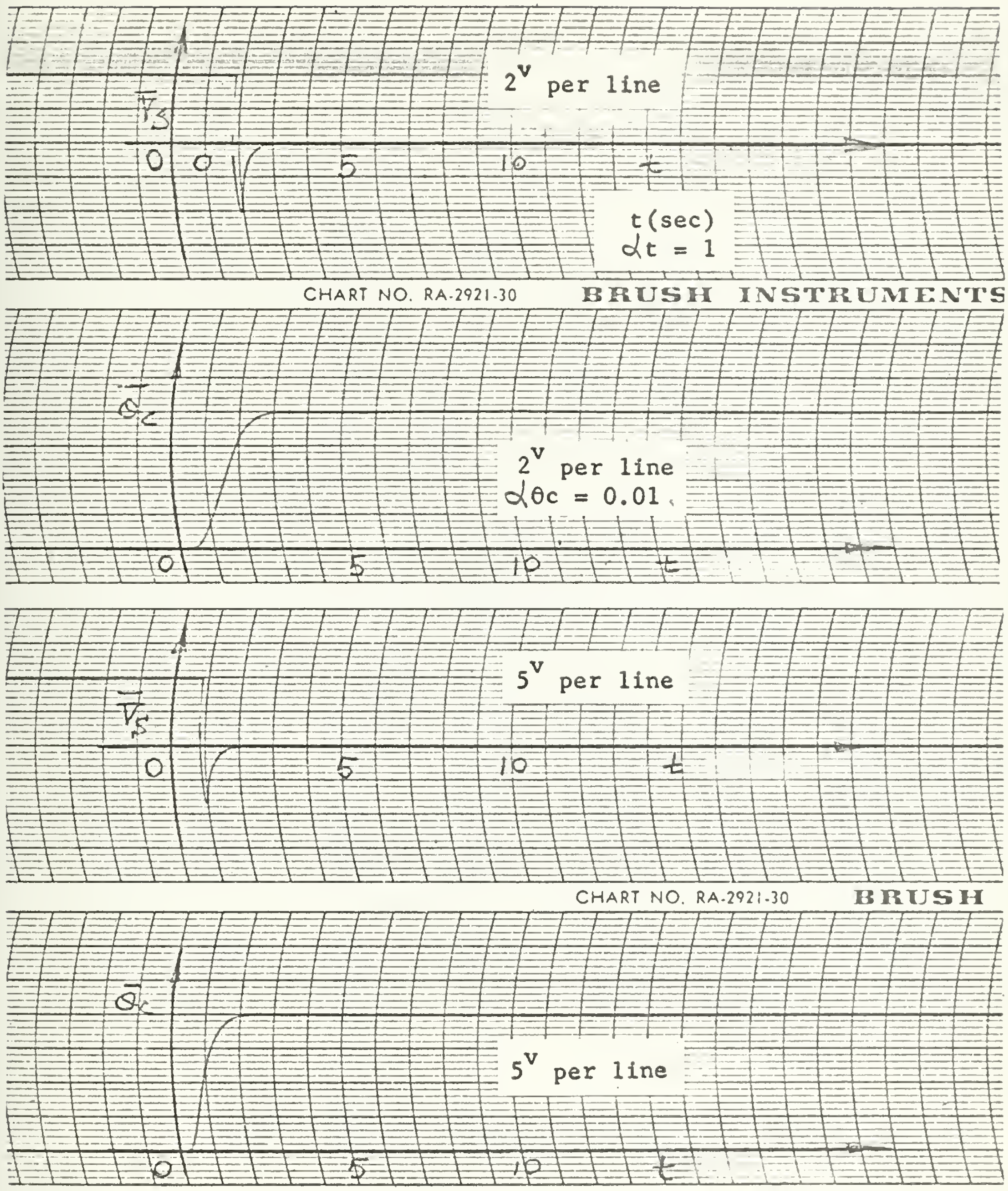


Fig. 4-21 Step Responses of a Saturated System with Different Level of Saturating Voltage.
(Computer Result $K = 255$, $V_S = 20^v$ and 50^v).

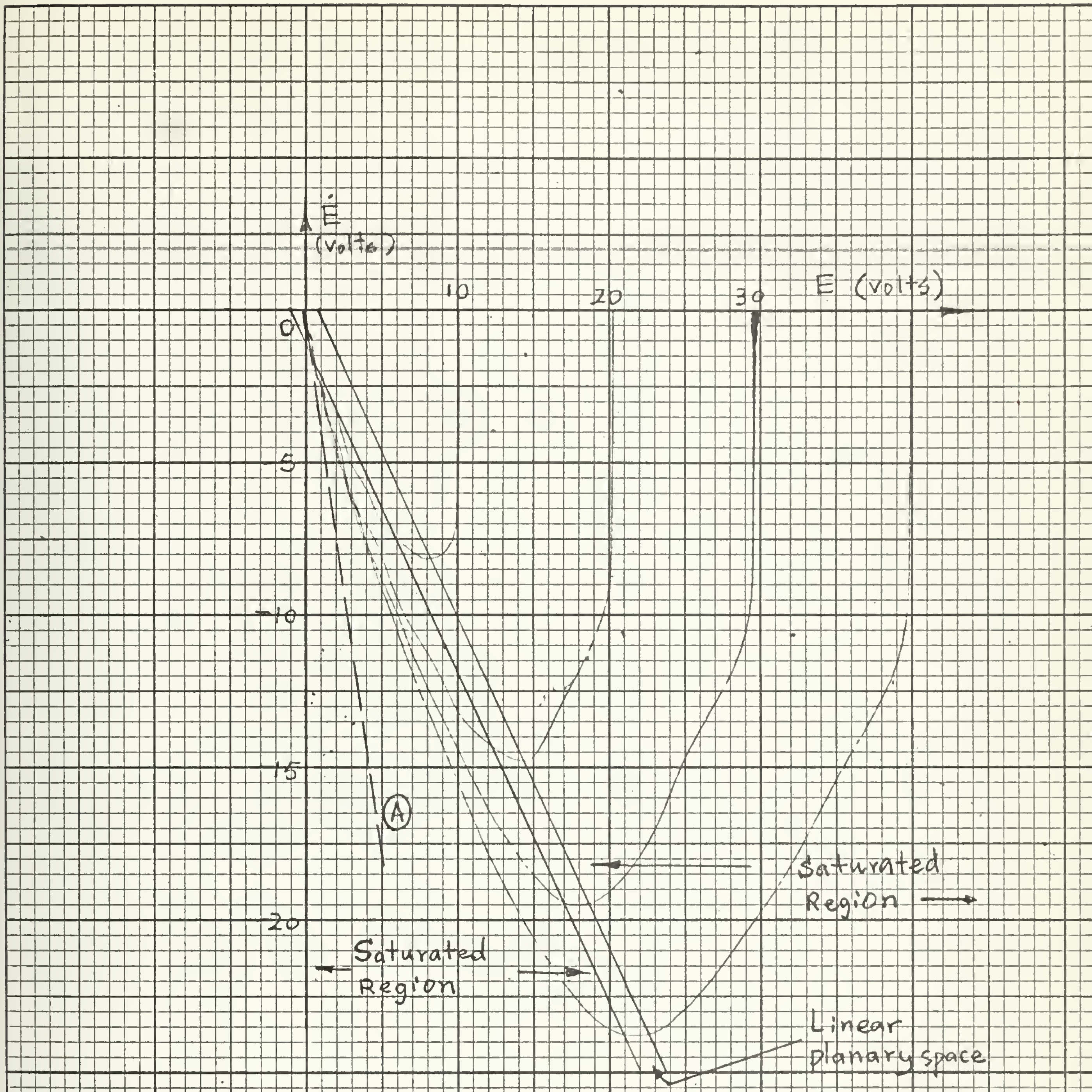


Fig. 4-22a E vs Ė Plot for a Third Order Saturated System with Step Inputs.
(Computer Result; K = 511 $V_s = 50^v$)
Ⓐ is the intersection of slow eigenplane with E vs Ė Plane.

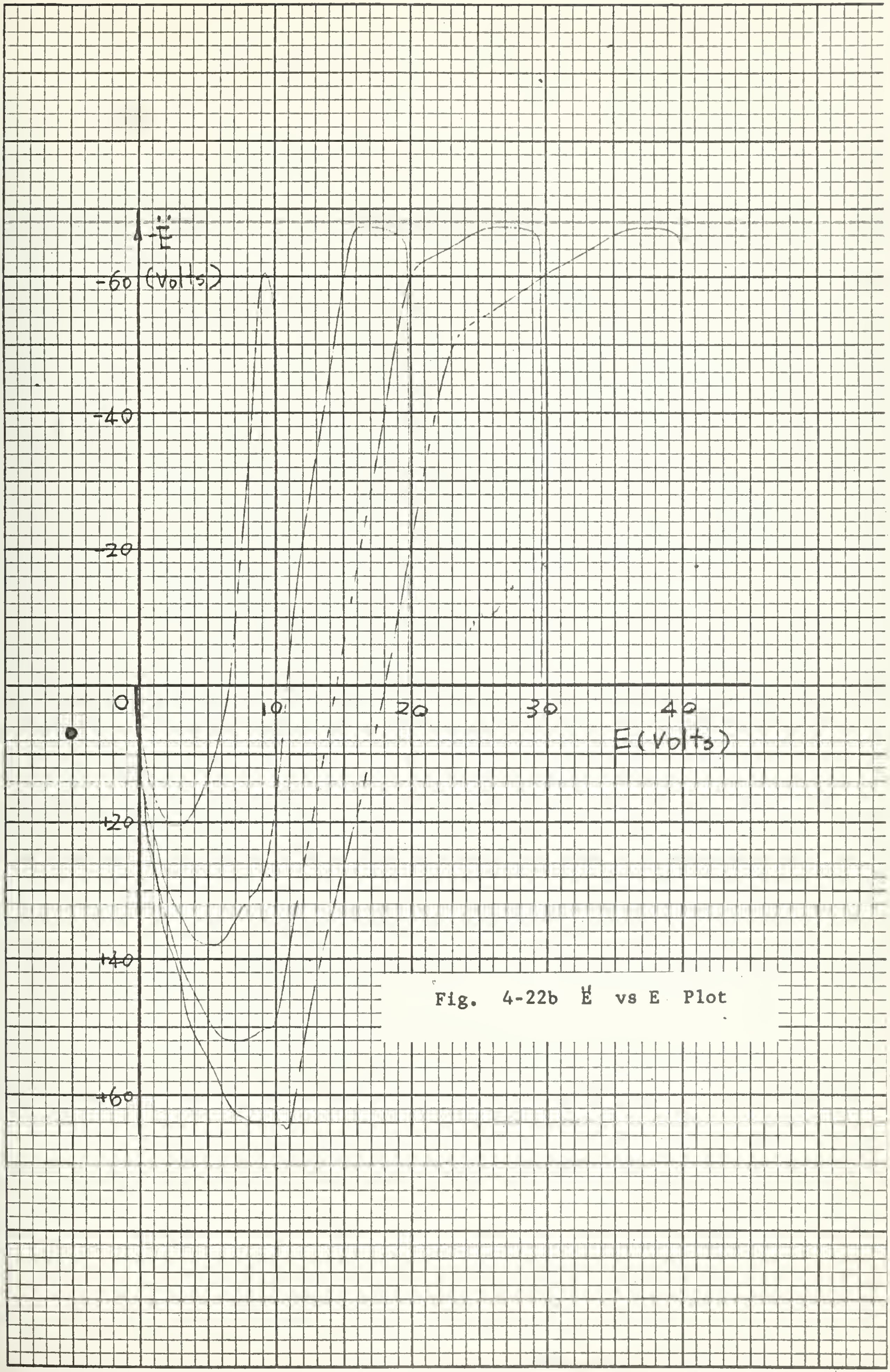


Fig. 4-22b $\ddot{E}$ vs E Plot

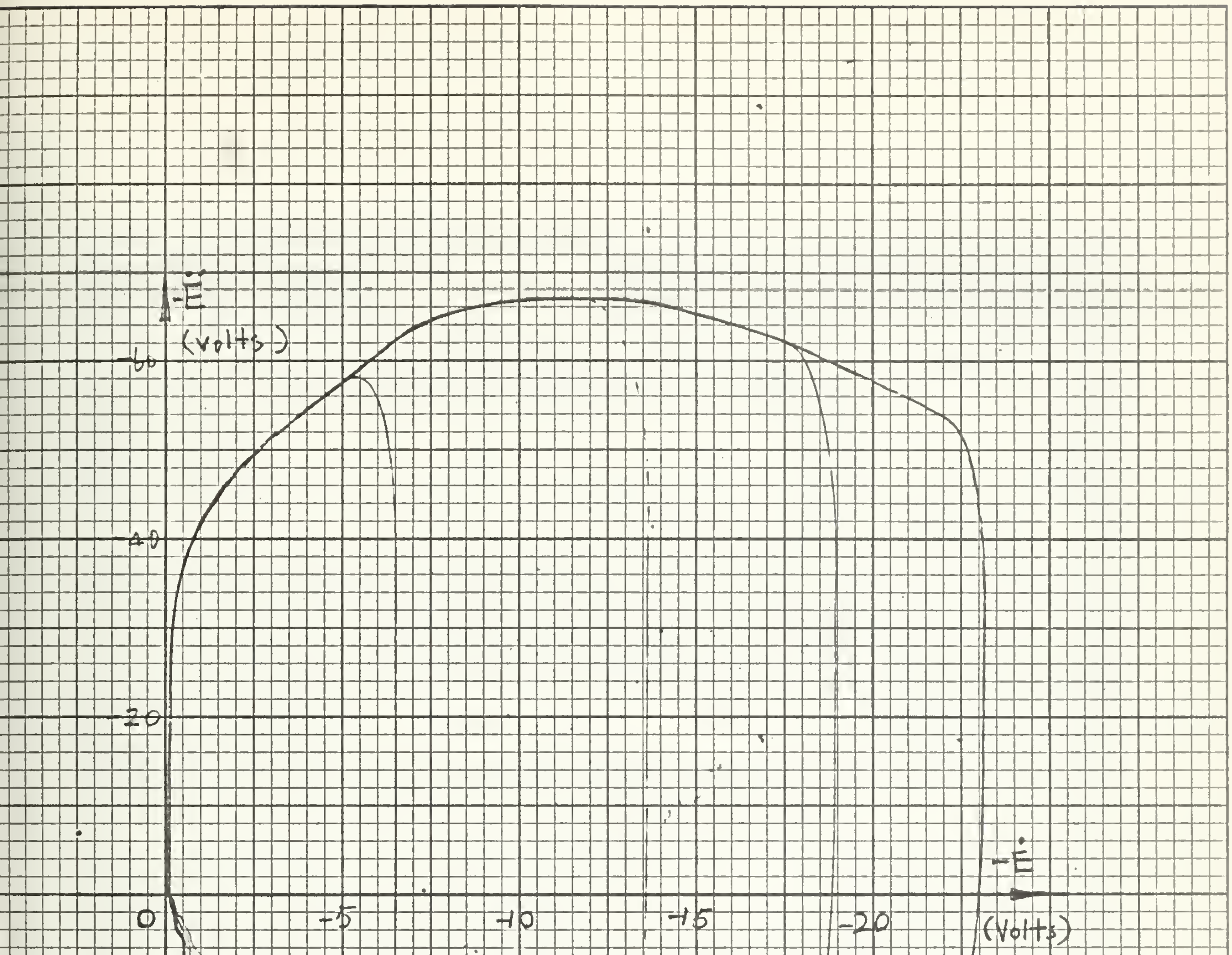


Fig. 4-22c $\ddot{E}$ vs $\dot{E}$ Plot

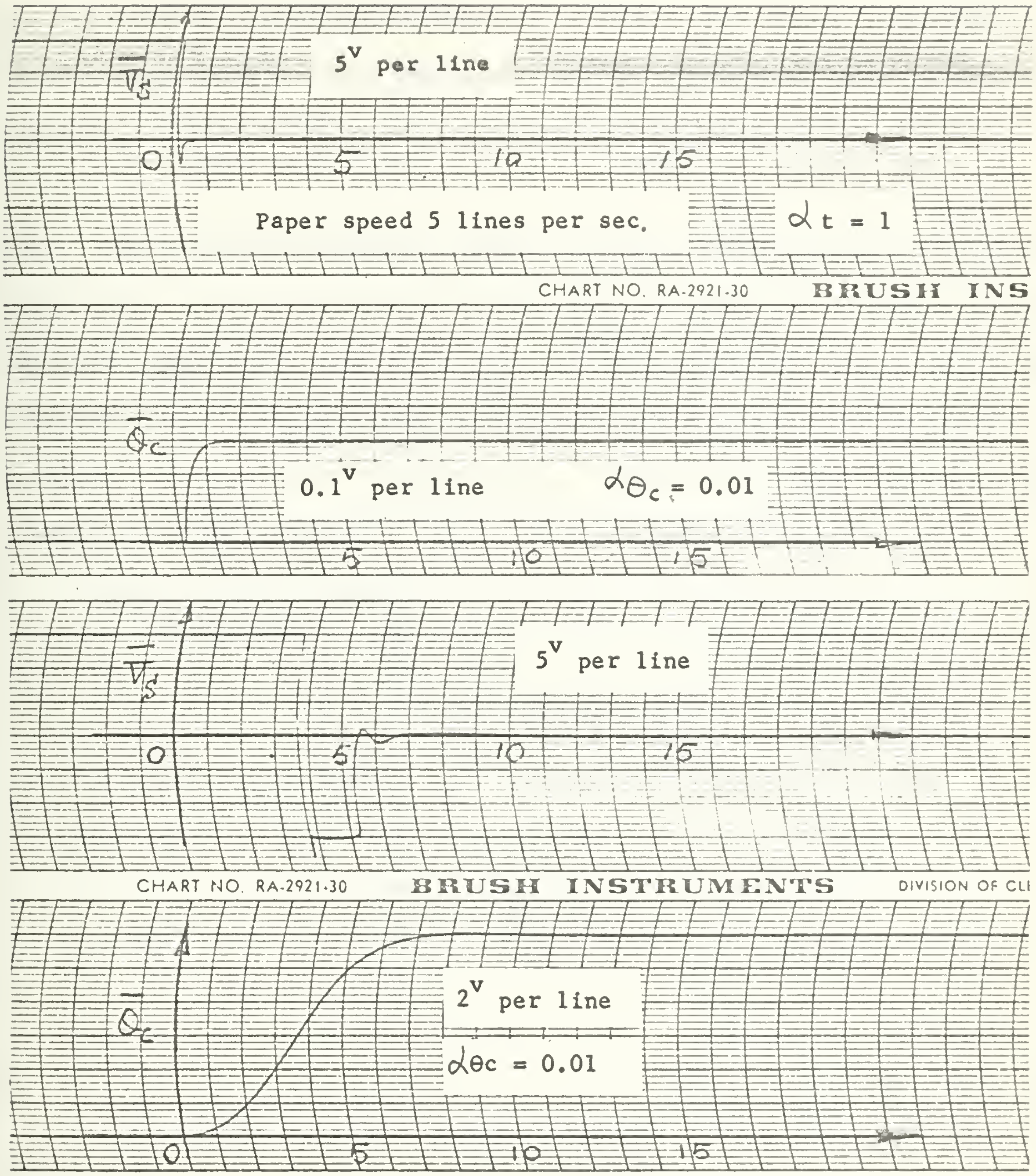


Fig. 4-23 Small and Large Step Responses of a High Gain Saturated System. (Computer Result K = 511).

SECTION IV - SATURATED SYSTEM WITH HIGH GAIN AND SMALL MOTOR LOAD TIME CONSTANT

For the system with $K = 511$, $P_1 = P_2 = 1$. The trajectory has an overshoot for large signal input as in Fig. 4-24. After selecting the maximum input step as 100^v, the system can be adjusted to have optimum response as in Fig. 4-25.

Note: The saturating amplifier output voltage reverses direction as the error nears zero. This is because the linear planary space has an intersection with the slow eigenplane in the right half of the error space. And also the velocity has a little overshoot at the end of the trajectory, but the effect upon the θ_c trajectory is negligible. This kind of situation is preferable in some practical systems.

Computer setting values:

$$\dddot{E} + (kk_a + 2)\ddot{E} + (kk_T + 1)\dot{E} + KE = 0 \qquad (4\text{-}111)$$

$$K = 511,\ K_a = 0.043,\ K_T = 0.373$$

$W_3 = 0.22$	$R_A = 10$	$R_3 = 1$	$a_3 = 0.22$
$W_4 = 38.25$	$R_A = 10$	$R_4 = 0.1$	$a_4 = 0.3825$
$W_6 = 1$	$C_B = 1.04$	$R_6 = 0.1$	$a_5 = 0.104$
$W_8 = 1$	$C_c = 1.04$	$R_8 = 0.1$	$a_6 = 0.104$

For 100^v step input optimum response readjustment

$$a_3 = 0.414$$

$$a_4 = 0.822$$

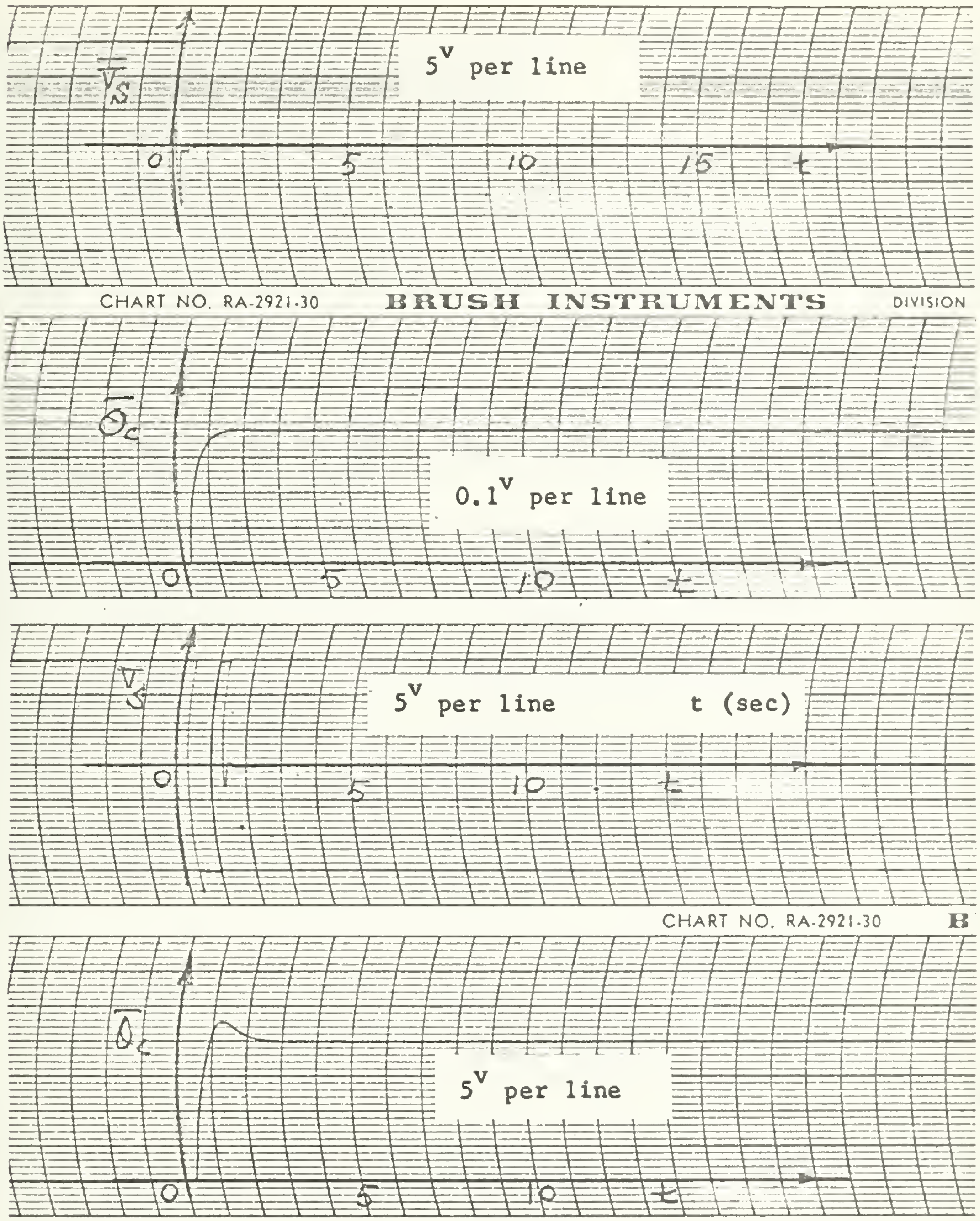


Fig. 4-24 Small and Large Step Responses for Critically Damped Saturated System.

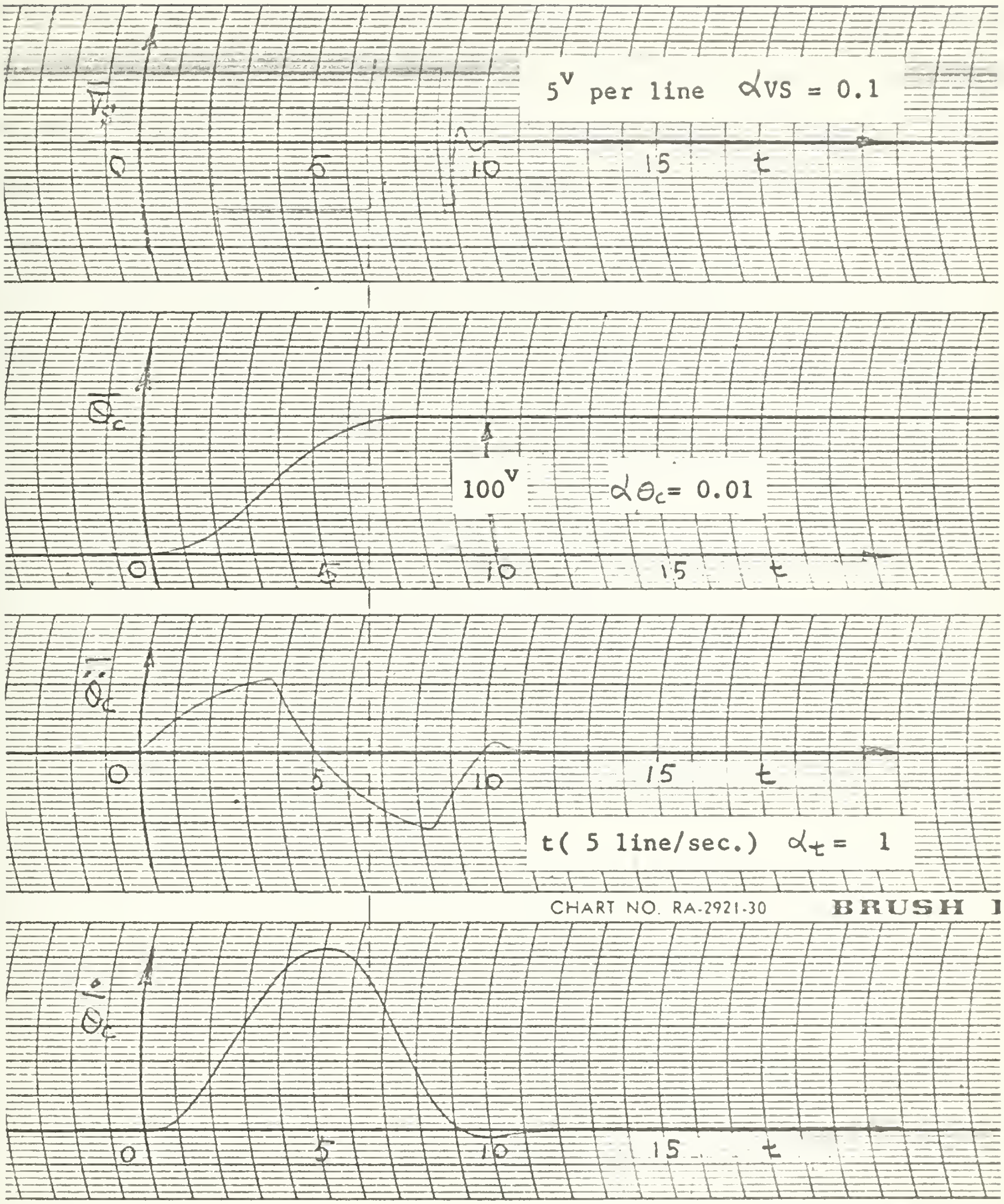


Fig. 4-25 Optimum Large Step Response after Adjusted. (Computer Result, $K = 511$, $P_1 = P_2 = 1$, $a_3 = 0.414$, $a_4 = 0.822$)

SECTION V - FREQUENCY RESPONSE

For low frequency sine wave input or high frequency but with small magnitude. Both the semilinear and saturated systems will not be saturated at the main amplifier output, because the system can follow the slow frequency sine wave input and the small signal will not saturate the system. Then the frequency response is just like those of linear systems. (Since all the systems here are using critical or overdamped feedback, then there will be no resonance peak in the frequency response curve).

For high frequency sine wave or large magnitude input the effect of saturation makes this kind of system sensitive to both magnitude and frequency. The computer results for a third order system (saturated) are shown in Fig. 4-26 and Fig. 4-27.

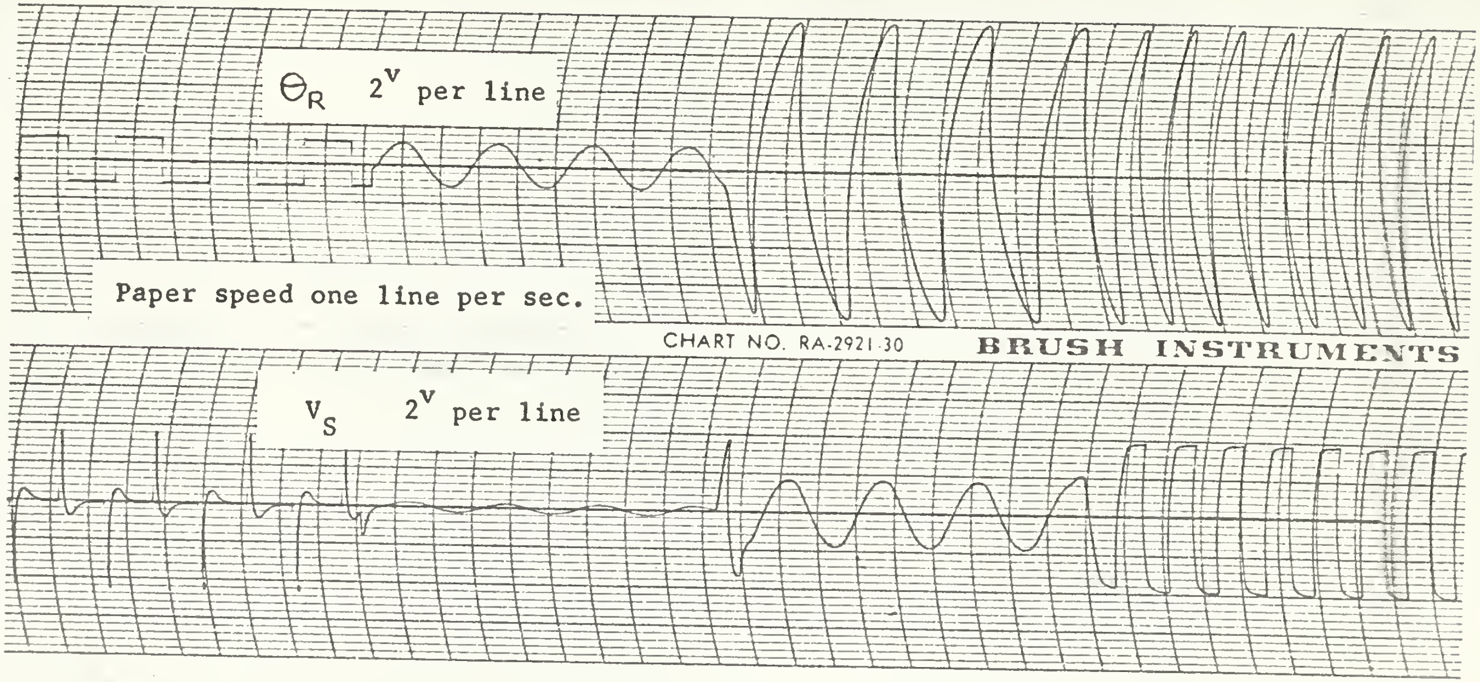


Fig. 4-26 Saturated Amplifier output Due to different Input Magnitude and Frequency

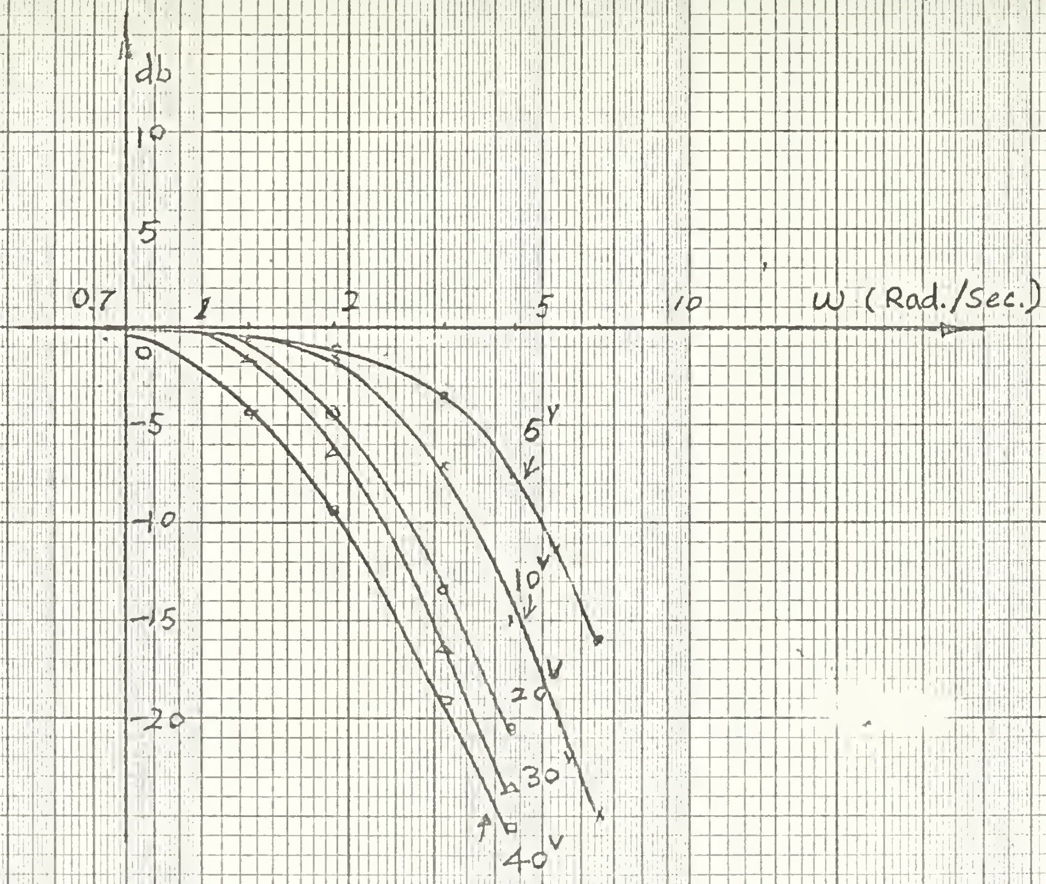


Fig. 4-27 Frequencey Response for a Third Order Saturated System. (Computer Result K = 255, $\bar{V}_s = 20^v$).

SECTION VI - COMPUTER STUDY OF FOURTH ORDER SYSTEMS

From the block diagram in Fig. 4-28, the open loop poles are:

$$P_1 = 2.3, \qquad P_2 = 2.5, \qquad P_3 = 2.7, \qquad \text{and} \quad K = 81$$

The compensated characteristic equation is

$$\dddot{E} + (KK_1 = 7.5)\,\dddot{E} + (KK_a + 18.7)\ddot{E} + (KK_T + 15.2)\dot{E} + KE = 0$$

For a semilinear system, the equation of linear space and slow eigenspace is:

$$\dddot{E} + (2.3 + 2.5 + 2.7)\ddot{E} + (2.3 \times 2.5 + 2.5 \times 2.7 + 2.3 \times 2.7)E$$
$$+ (2.3 \times 2.5 \times 2.7)E = 0$$

i.e.

$$\dddot{E} + 7.5\ddot{E} + 18.7\dot{E} + 15.52E = 0$$

For $r_4 = 5.21$, then

$$K_1 = 0.0644$$

$$K_a = 0.485$$

$$K_T = 1.204$$

The equations for computer setup are:

$$[-\bar{V}_S] = [w_1\bar{\theta}_R - w_2\bar{\theta}_c - w_3\ddot{\bar{\theta}}_c - w_4\dot{\bar{\theta}}_c - w_5\dot{\bar{\theta}}_c]$$

$$[+\bar{J}] = \frac{w_6}{p + w_7}[-\bar{V}_S]$$

$$[-\bar{Q}] = \frac{w_8}{p - w_9}[+\bar{J}]$$

$$[+\dot{\bar{\theta}}_c] = \frac{w_{10}}{p + w_{11}}[-\bar{Q}]$$

$$[-\bar{\theta}_c] = \frac{1}{p} w_{12}[+\dot{\bar{\theta}}_c]$$

$$[+\ddot{\bar{\theta}}_c] = p\,w_{14}[\dot{\bar{\theta}}_c]$$

$$[-\dddot{\bar{\theta}}_c] = p\,w_{16}[+\ddot{\bar{\theta}}_c]$$

Where

$$w_1 = w_2 = \frac{K\alpha e_R}{\alpha V_S},\ w_3 = \frac{KK_1\alpha\ddot{\theta}_c}{\alpha V_S},\ w_4 = \frac{KK_a\alpha\ddot{\theta}_c}{\alpha V_S},\ w_5 = \frac{KK_T\alpha\dot{\theta}_c}{\alpha V_S},\ w_6 = \frac{\alpha V_S}{\alpha_J\alpha_t},\ w_7 = \frac{2.3}{\alpha_t}$$

$$w_8 = \frac{\alpha_J}{\alpha_Q\alpha_t},\ w_9 = \frac{2.5}{\alpha_t},\ w_{10} = \frac{\alpha_Q}{\alpha\dot{\theta}_c\alpha_t},\ w_{11} = \frac{2.7}{\alpha_t},\ w_{12} = \frac{\alpha\dot{\theta}_c}{\alpha_{\theta_c}\alpha_t},\ w_{14} = \frac{\alpha\ddot{\theta}_c}{\alpha\dot{\theta}_c},\ w_{16} = \frac{\alpha\dddot{\theta}_c}{\alpha\ddot{\theta}_c}$$

The calculated values for each element in the computer setting are in Table 4-3. The diagram of analog computer setup is in Fig. 4-29.

The computer results of this setup have not been taken, the suggested further investigations are as follows:

(a) Semilinear system with real open loop poles (as the computer set up in above pages).

(b) Saturated system with real open loop poles (increase the gain and recalculate the feedback signals in the above system.)

(c) Semilinear system with complex open loop poles (change the coefficients of the uncompensated characteristic equation).

(d) Saturated system with complex open loop poles (as in (b)).

(e) Semilinear and saturated system when $\dddot{\theta}_c$ signal is not available.

(f) Switching techniques in discontinuous linear, semilinear or saturated system for taking out a pair of complex poles.

The investigation for ramp input is given in Chapter V.

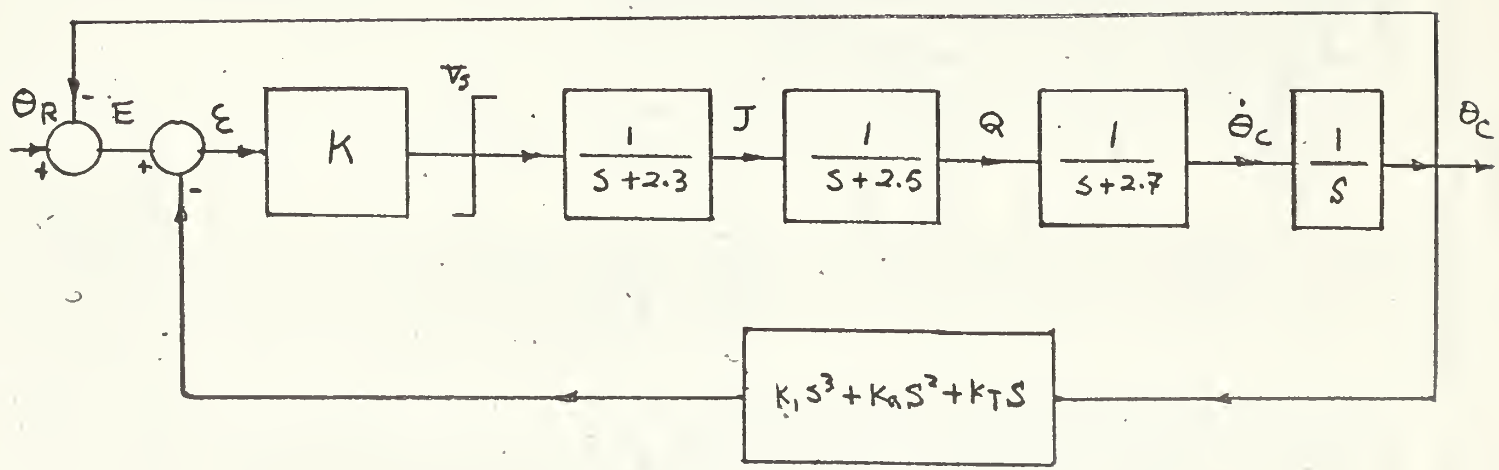


Fig. 4-28 Block Diagram for a Fourth Order Feedback Control System with Saturated Main Amplifier.

TABLE 4-3

Computer setting values for Fourth Order semilinear system

	$W_1=8.1$		$R_1=1$	$a_1=0.81$
W_1 =	$W_2=8.1$		$R_2=1$	$a_2=0.81$
	$W_3=0.521$	$R_A=10$	$R_3=1$	$a_3=0.0521$
	$W_4=3.93$		$R_4=1$	$a_4=0.393$
	$W_5=9.74$		$R_5=1$	$a_5=0.974$
	$W_6=1$	$C_B=1.02$	$R_6=0.5$	$a_6=0.51$
	$W_7=2.3$		$R_7=0.2$	$a_7=0.469$
	$W_8=2$	$C_C=1.02$	$R_8=0.2$	$a_8=0.408$
	$W_9=2.5$		$R_9=0.2$	$a_9=0.51$
	$W_{10}=5$	$C_d=1.02$	$R_{10}=0.1$	$a_{10}=0.51$
	$W_{11}=2.7$		$R_{11}=.1$	$a_{11}=0.275$
	$W_{12}=1$	$C_E=1.04$	$R_{12}=0.5$	$a_{12}=0.52$
	$W_{13}=1$	$R_F=1$	$R_{13}=0.5$	$a_{13}=0.5$
	$W_{14}=1$	$R=10$ $C_r=0.44$	$R_{14}=0.1=r_r$	$a_{14}=0.227$
	$W_{15}=1$	$R=1$	$R_{15}=15$	$a_{15}=0.5$
	$W_{16}=1$	$R_{16}=10$ $C=0.44$	$R_{16}=0.1=r_r$	$a_{16}=0.227$

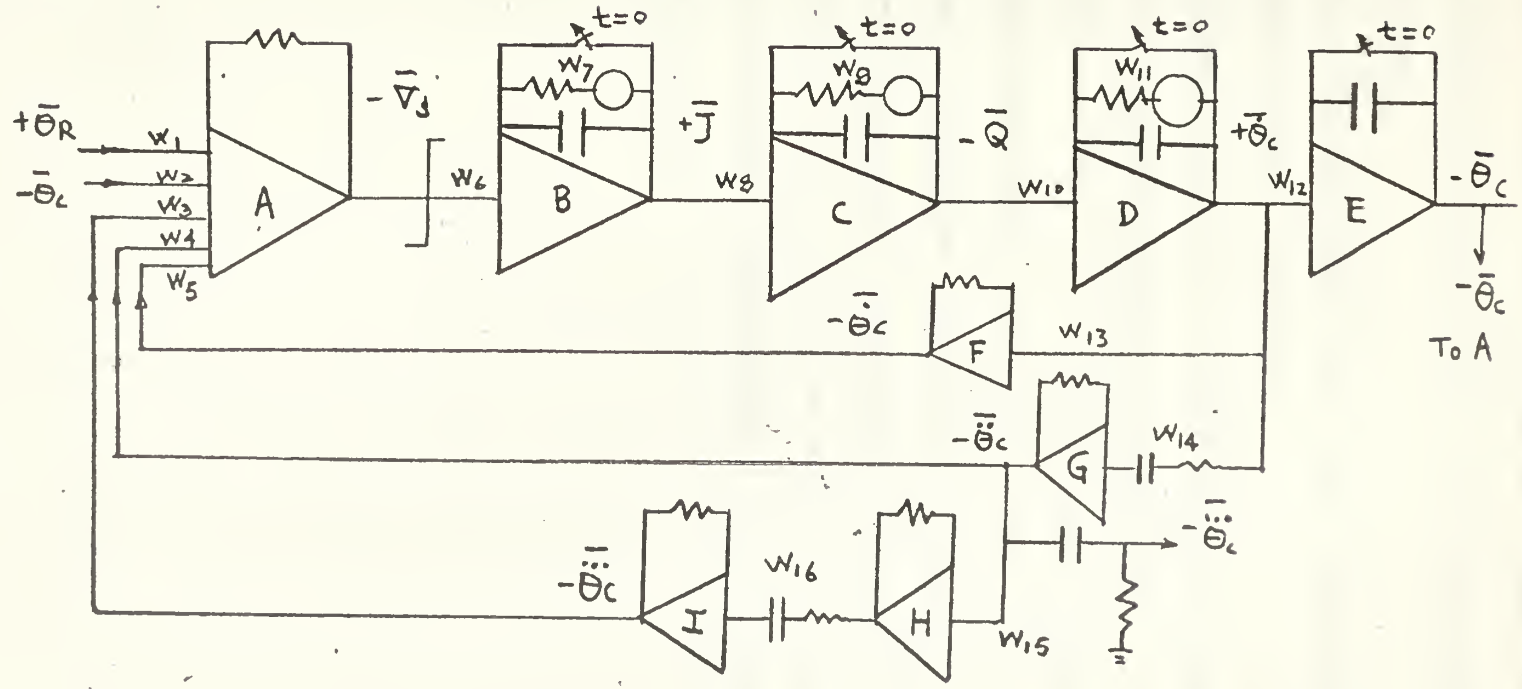


Fig. 4-29 Analog Computer Setup for a Fourth Order Semilinear System

General Conclusions

For low gain systems, the linear space is large, the system can be regarded as a completely linear system, and the fast eigenplane switching method can be used.

For low or high gain systems with a small motor-load time constant, the semilinear design theory is applicable.

If a high gain system has a large motor-load time constant then the design theory of saturated systems can be applied.

If a high gain system has large complex open loop poles, then the approximate semilinear design theory can be used; and if the complex poles are small, then the saturated design theory will give a better result.

Since the maximum ability of a system is decided by the system parameters then the first thing to do is to analyze the given system to determine in which of the above classifications it belongs, and then apply the proper design procedures.

CHAPTER V - RAMP RESPONSE ANALYSIS AND DESIGN OF DISCONTINUOUS SYSTEMS.

SUMMARY

An investigation is made to design a type I system to have optimum response for both step and ramp inputs. High gain of main amplifier is used to reduce the lagging error due to a ramp input, underdamped trajectory provides fast response and heavy damped feedback signals are used to make the system to be dead beat, or nearly dead beat. Switching circuits make the trajectory of ramp response to reach the steady state point and stop there.

SECTION I - INTRODUCTION AND BASIC THEORY.

In the design of a type one system, in order to make the steady state lagging error small, a system with a high gain and small motor-load open loop pole is preferable. Unfortunately such a system will be badly underdamped. If the feedback signals are used to make the system meet the transient response requirement, then the steady state lagging error will be increased.

In this investigation, by phase plane analysis, for second order systems by plotting the Isoclines for both the underdamped and heavily overdamped case, there will be one trajectory in the overdamped case just passing through the steady state lagging error point (focal point) of the underdamped case, (for a physical system this is the best point one can get), and also there is an intersection with the underdamped trajectory. Therefore by using a switching computer the heavy damping is only applied in an interval between these two points. The system will have optimum response for ramp input.

In third or higher order systems since the trajectories are affected by ramp input only by causing the translation of the coordinate along the error axis, a modified switching circuit will make the system have an optimum response to ramp input.

Since the die-out trajectory due to ramp input is not the same form as for the step input, then the investigation of the die-out trajectory is necessary to design the switching computer.

In the second order system, the phase plane method is used, and in the third order system phase space analysis is used as in Chapter I and II.

SECTION II-PHASE PLANE ANALYSIS FOR SECOND ORDER SYSTEM. (WITH RAMP INPUT)

From Fig. 5-1 the general characteristic equation can be derived as

$$\ddot{\theta}_c + a\dot{\theta}_c = KE - KK_T\dot{\theta}_c \tag{5-1}$$

$$-\ddot{E} + (a+KK_T)(\dot{\theta}_R - \dot{E}) = KE \tag{5-2}$$

$$\ddot{E} + (a+KK_T)\dot{E} + KE = (a+KK_T)\dot{\theta}_R = (a+KK_T)\omega_i \tag{5-3}$$

Let $N = \dfrac{d\dot{E}/\omega_n}{dE}$, $\omega_n^2 = K$

Then

$$\frac{\dot{E}}{\omega_n} = \frac{\dfrac{KK_T+a}{\omega_n^2}\omega_i - E}{N + \dfrac{KK_T+a}{\omega_n}} = \frac{(K_T + \frac{a}{\omega_n^2})\omega_i - E}{N + \dfrac{KK_T+a}{\omega_n}} \tag{5-4}$$

For underdamped case, assume $\omega_i = 1$ rad./sec., $K_T = 0$; $a = 2.6$, $K = 64$

Then $\dfrac{\dot{E}}{\omega_n} = \dfrac{0.0406 - E}{N + 0.325}$ (equation of straight lines)

$\omega_i/\omega_n = 0.125$

Setting $E = 0$ and changing the value of N , then get the following table, and the Isocline plot in Fig. 5-2.

For E = 0,

N	0	-1	-0.5	-3	+1	+3	+0.5
$\dot{E}/\omega_n$	+0.125	-0.064	-0.232	-0.0152	0.0306	0.0122	0.0492

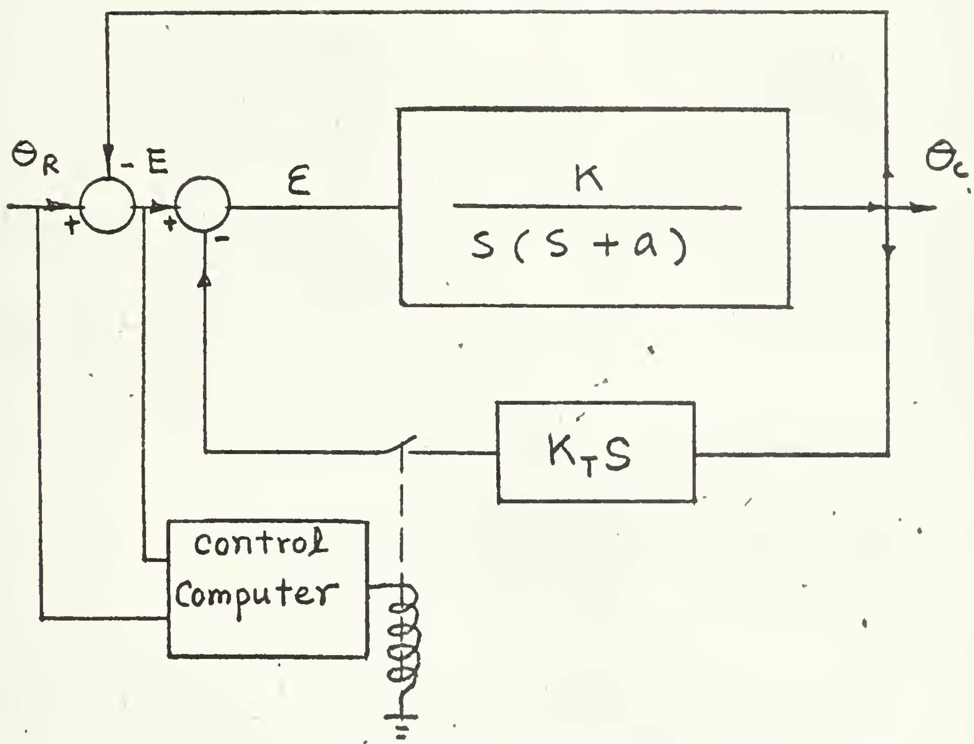
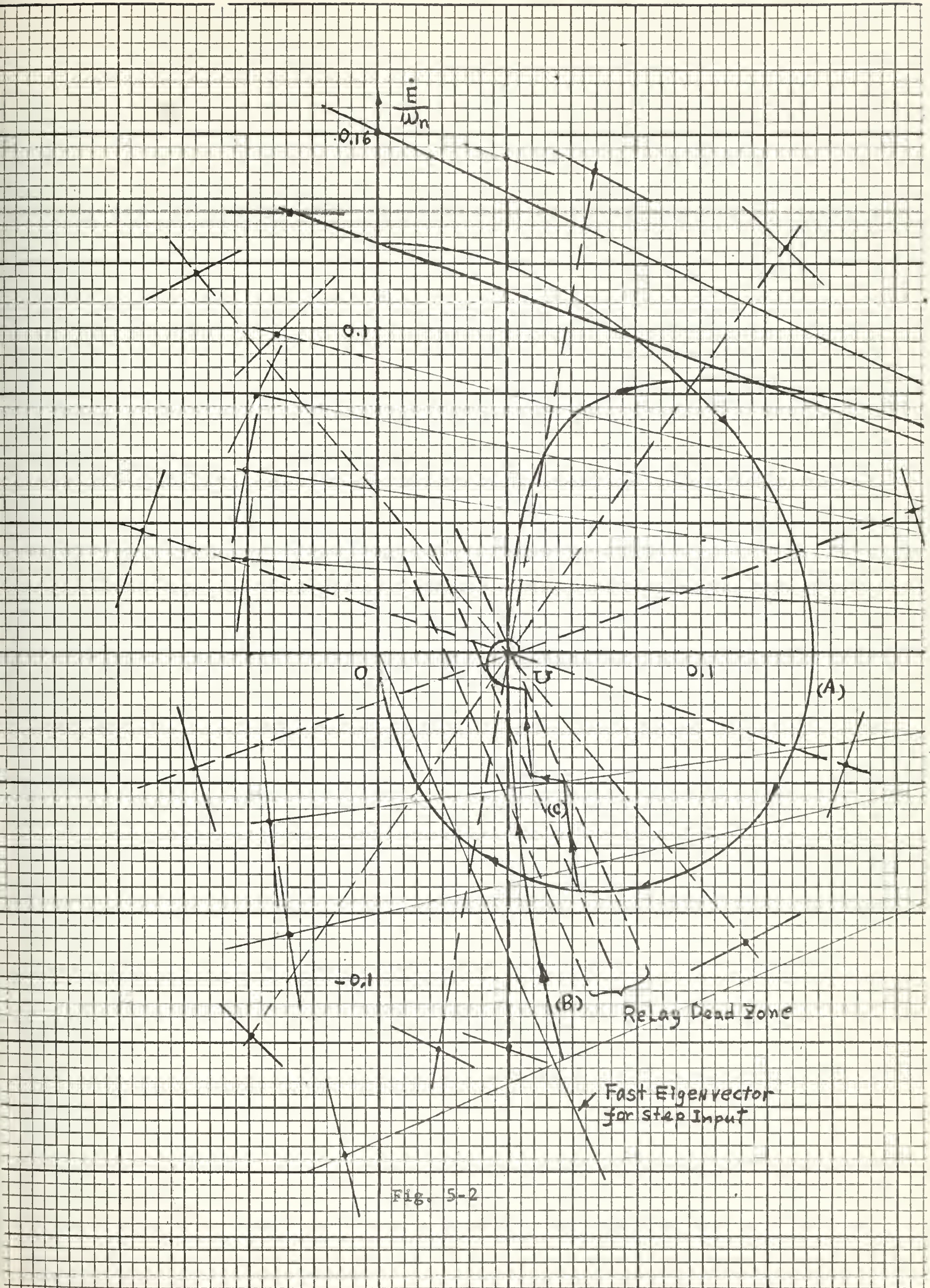
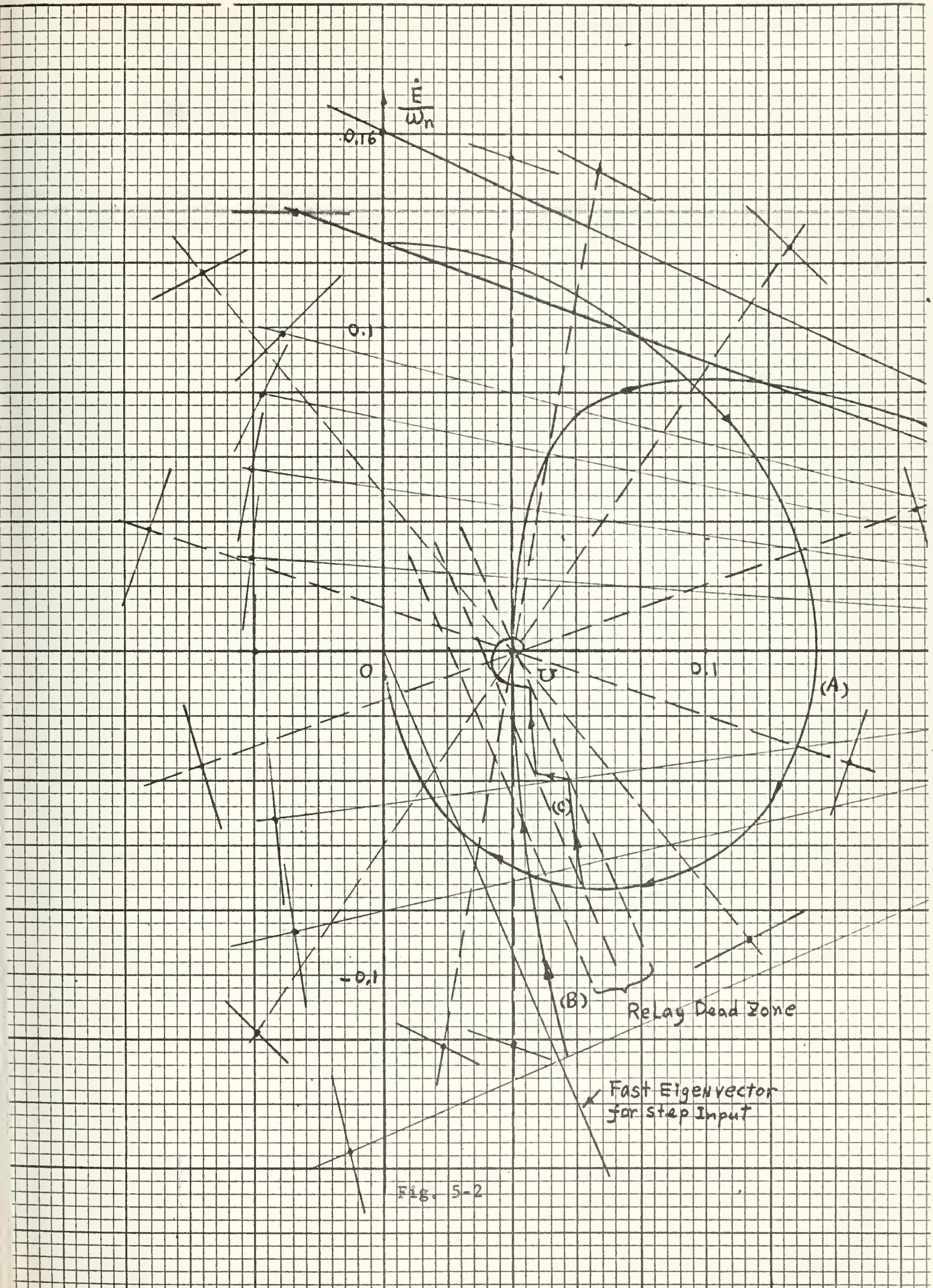


Fig. 5-1 Block Diagram of a Second Order System with Discontinuous Tachometer Feedback.



$\frac{\dot{E}}{\omega_n}$
0.16
0.1
O
U
0.1
(A)
(C)
-0.1
(B)
Relay Dead Zone
Fast Eigenvector for Step Input

Fig. 5-2



$\frac{\dot{E}}{\omega_n}$
0.16
0.1
O
U
0.1
(A)
(C)
-0.1
(B)
Relay Dead Zone
Fast Eigenvector for Step Input

Fig. 5-2

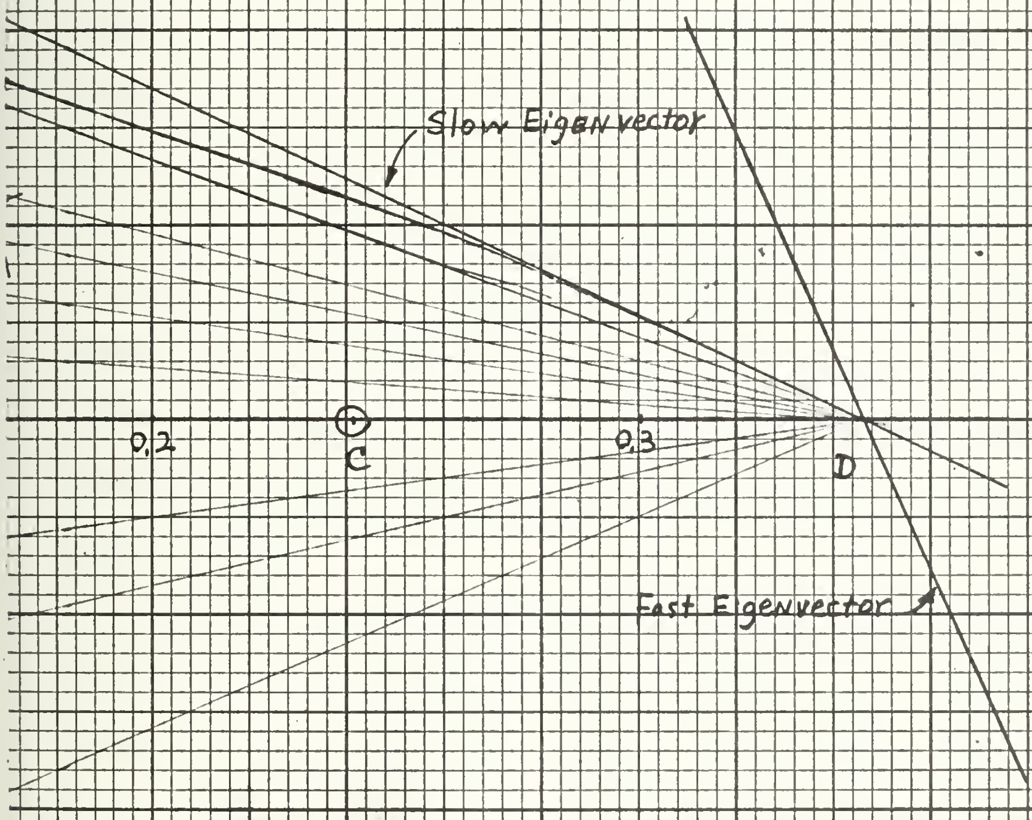


Fig. 5-2 Phase Plane Plot of the Underdamped and Overdamped Trajectories.
(a) Under damped trajectory
(b) Overdamped trajectory
(c) Trajectory in Relay Dead Zone
(point c is the lagging error of critically damped system)

For overdamped system by choosing $K_T = 0.3$, then

From Fig. (5-4) $$\frac{\dot{E}}{\omega_n} = \frac{0.3406 - E}{N + 2.725} \qquad (5\text{-}5)$$

The table of Isocline intersections with axis is as follows:

For $E = 0$,

N	0	+1	+2	+5	110	-5	-7	-2.288
$\dot{E}/\omega_n$	0.125	0.0915	0.072	0.0506	-.0268	-0.15	-0.08	+0.78

The plot is in Fig. 5-2.

NOTE: Fig. 5-2 can be gotten from Fig. 2-3b in Chapter II by moving the origin. It indicates that by using discontinuous damping if the switching device can follow the die-out trajectory then the state point can be switched to pass any point on the E axis on the left side of the fast eigenvector, but the points keep their position in the steady state (ramp) are only the two focal points. For example if the state point is following an overdamped trajectory and intersecting the E axis at point N in Fig. 5-3, the state point will continue to move toward those focal points, because both the trajectories are going to the right side no matter whether the control switch is open or closed.

If the switching line has an error to the right of the optimum switching line or to the left, (this means that the switch operated earlier or later than the proper control signal), then a steady state error will be caused. As indicated in Fig. 5-4a, the switch operated earlier, the final error is to the right of the point $\mho$, this means the lagging error will be larger than the underdamped case, but the error is confined to the switch dead zone.

In Fig. 5-4b, the switch is operated later, then trajectory will spiral around the point $\mho$, and the final steady state error is zero. Therefore, the best way of finding the switching line is to make its intersection at the left of the point $\mho$ as in Fig. 5-5, and let the point $\mho$ stay in

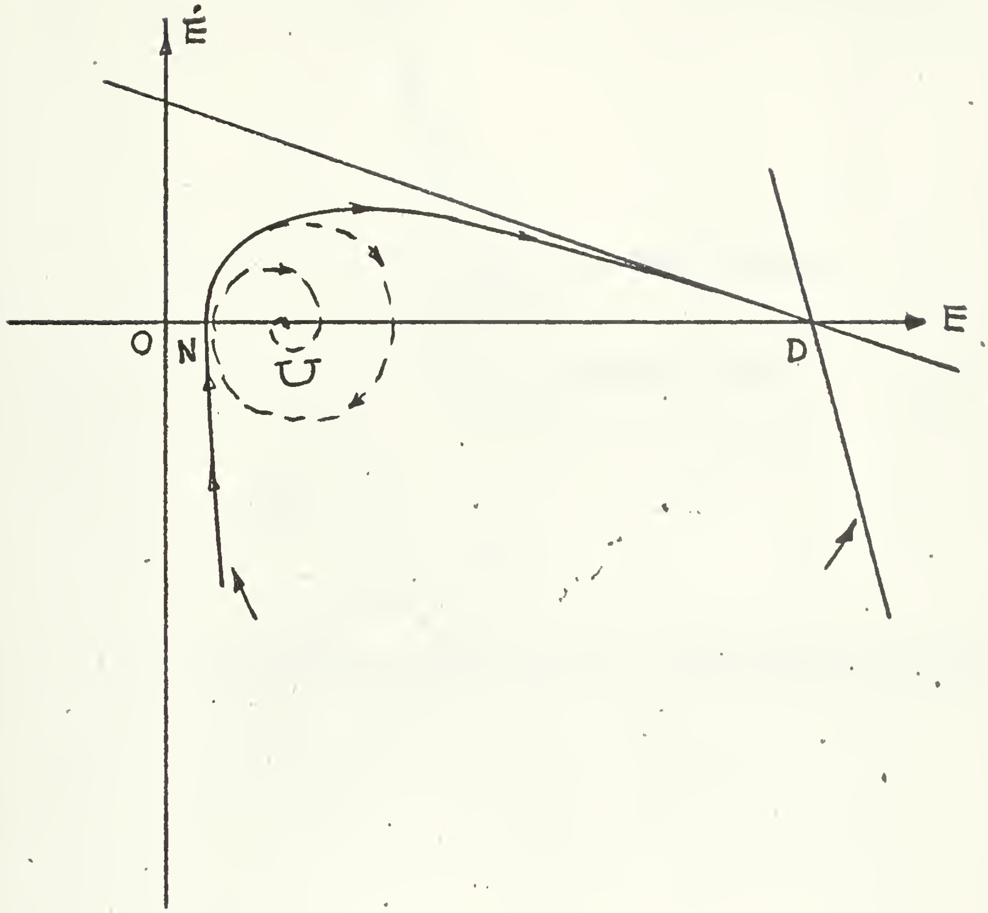
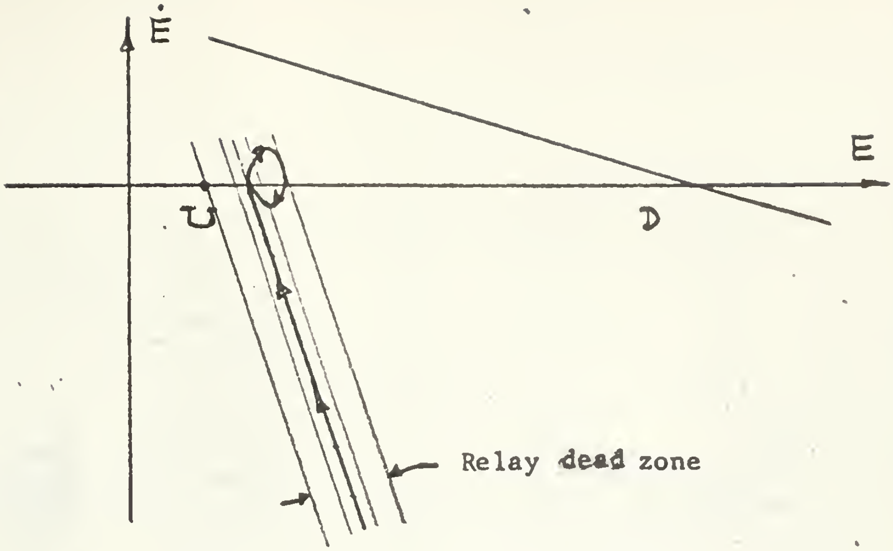
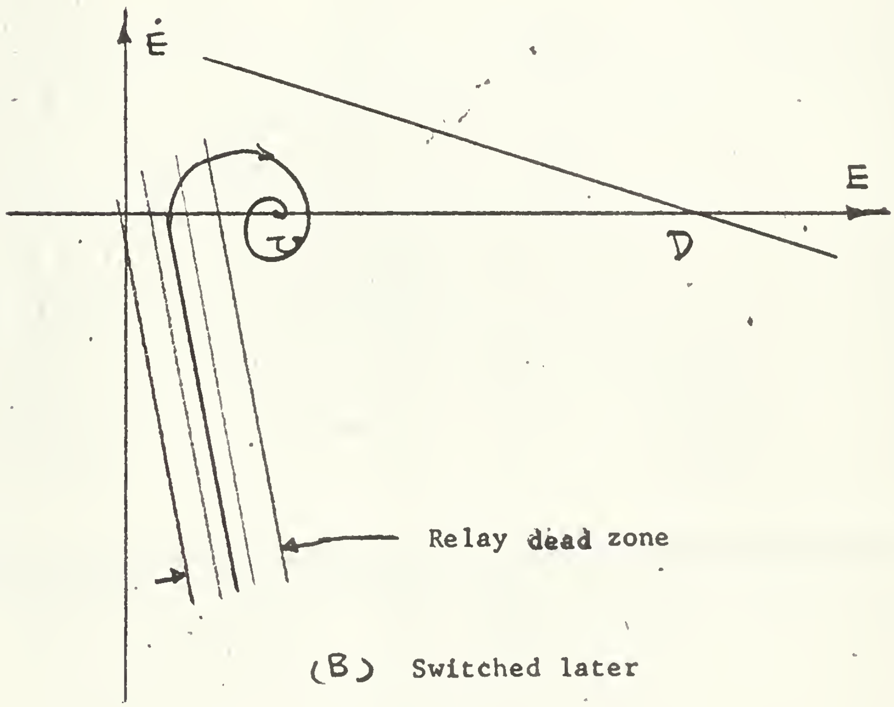


Fig. 5-3 Illustration of the Trajectories to the left side of the Underdamped steady state Point U .



(A) Switched Earlier



(B) Switched later

Fig. 5-4 The Steady State Error Caused by Switching early or late.

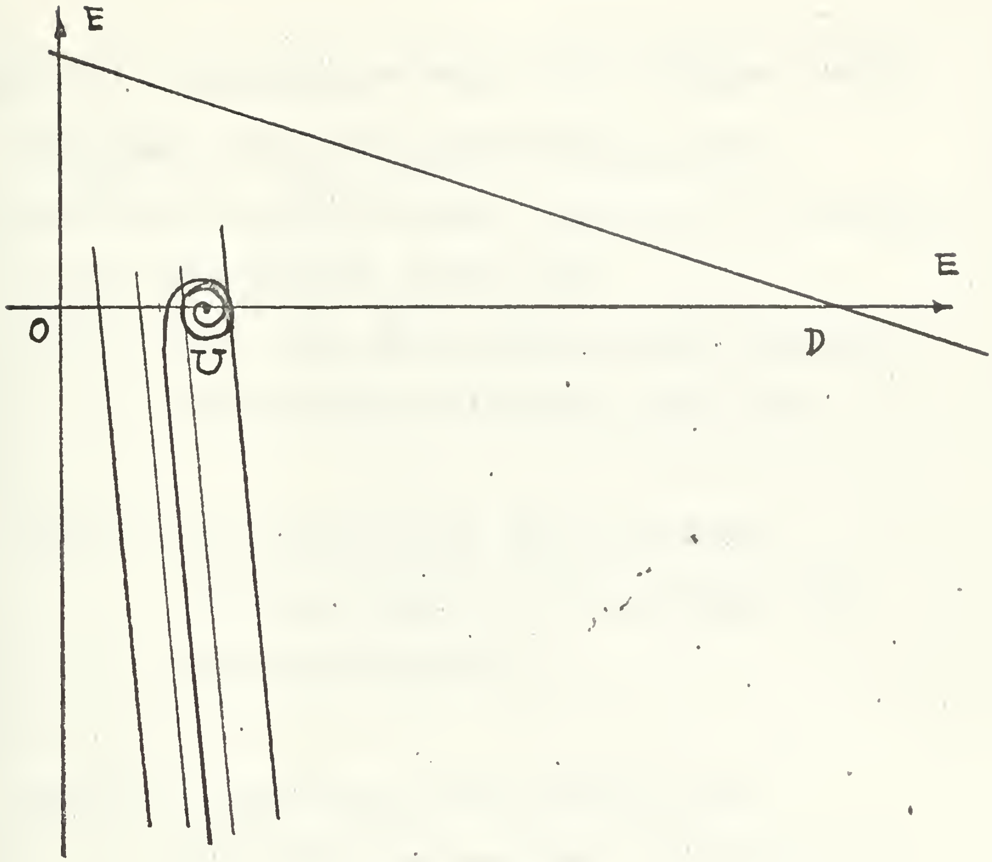


Fig. 5-5 The best position for Switching for Ramp Input

the dead zone of the relay.

If the width of dead zone of the relay can be neglected, then this modification as in Fig. 5-6 is not necessary.

SECTION III - DISCUSSION AND DESIGN ABOUT SWITCHING COMPUTER.

Switching lines and the relay dead zone effect:

I. Unmodified switching line (fast eigenvector for step input)

A. For same magnitude of ramp input

CASE I - Focal point is at outside of relay dead zone.
The trajectory is sketched in Fig. 5-6a.

CASE II - Focal point at the edge of dead zone.
Overdamped trajectory passing through focal point as in Fig. 5-6b.

CASE III - Focal point outside the dead zone.
Relay with wide dead zone. The trajectory is sketched in Fig. 5-6c.

CASE IV - Focal point inside the dead zone.
Slope of switching line too small (less negative) or dead zone too wide. The trajectory will oscillate as in Fig. 5-6d.

B. Same switching line and dead zone but different magnitude of ramp input.

CASE I - Small ramp input: Since the lagging error corresponding to the uncompensated system is

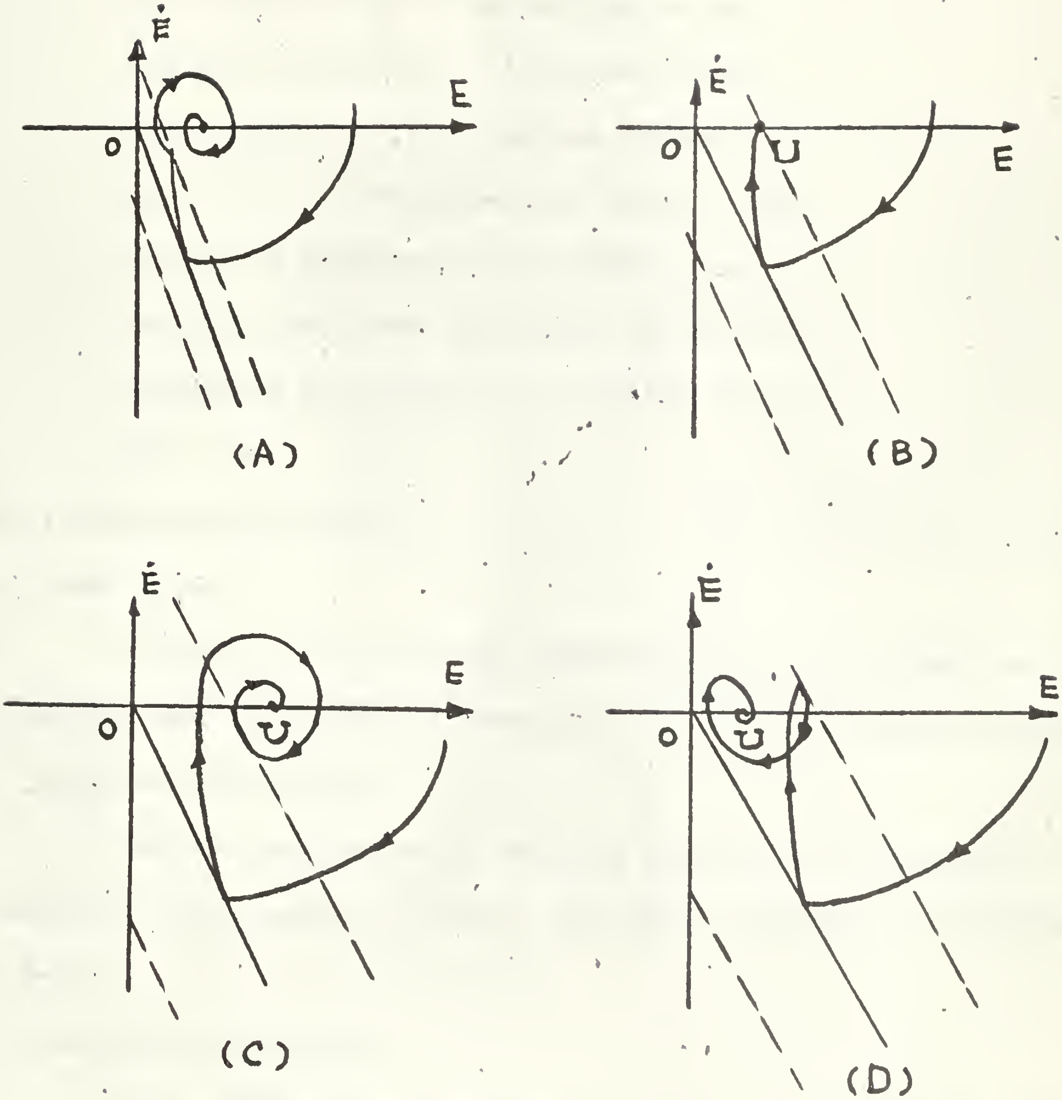


Fig. 5-6 Sketch Trajectories for same Magnitude of Ramp Input but with Different Relay Dead Zone and Focal Point Location.

small, then the whole trajectory may stay in the dead zone of the relay as sketched in Fig. 5-7a.

CASE II - Large ramp input: The underdamped stable point (focal point) is in the relay dead zone, the die-out trajectory may or may not hit this point, a little oscillation will occur near this point as sketched in Fig. 5-7b, but when the ramp input is very large, the location of the stable point is out side the linear zone, then the die-out trajectory will chatter or oscillate as in Fig. 5-7c.

II. Modified Switching Line No. I. $\{ u = (-\frac{a\omega_i}{K} + E)r + \dot{E} = 0]$

A. Ideal relay:

In this case the die-out trajectory will pass through the focal point and stay there without chattering like that sketched in Fig.5-8a.

B. Relay with dead zone:

The die-out trajectory chatters down and a little oscillation near the focal point will occur. The typical trajectory is sketched in Fig. 5-8b.

C. Wide relay dead zone.

The switching line (the center line) of the relay must shift to the left of the focal point in order to reduce the oscillations near that point, and the chattering situation will become severe as indicated in Fig. 5-9.

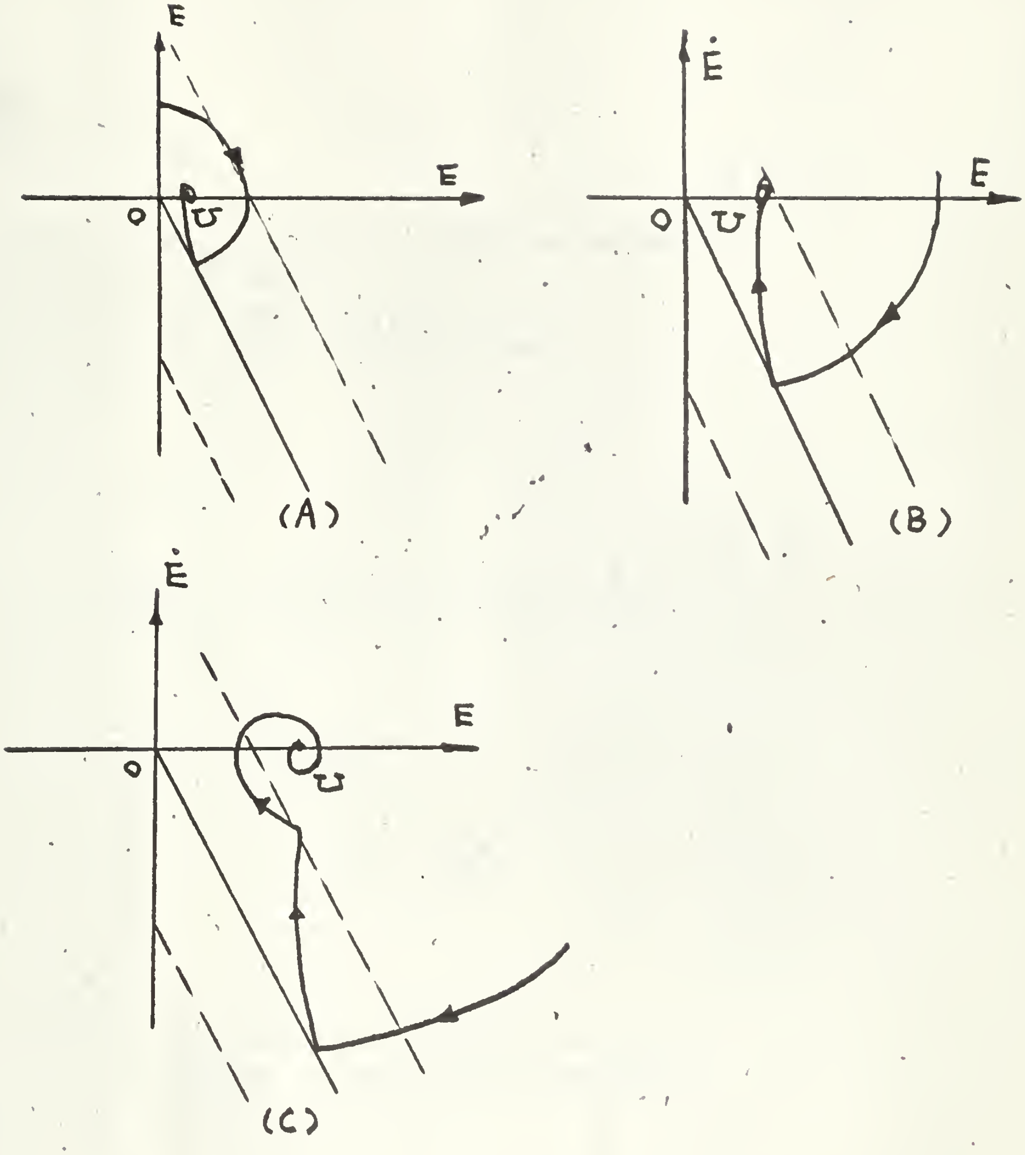


Fig. 5-7 Trajectories of Ramp Input Response by Using Same Switching Line and Relay.
(a) Small Ramp Input
(b) & (c) Large Ramp Input

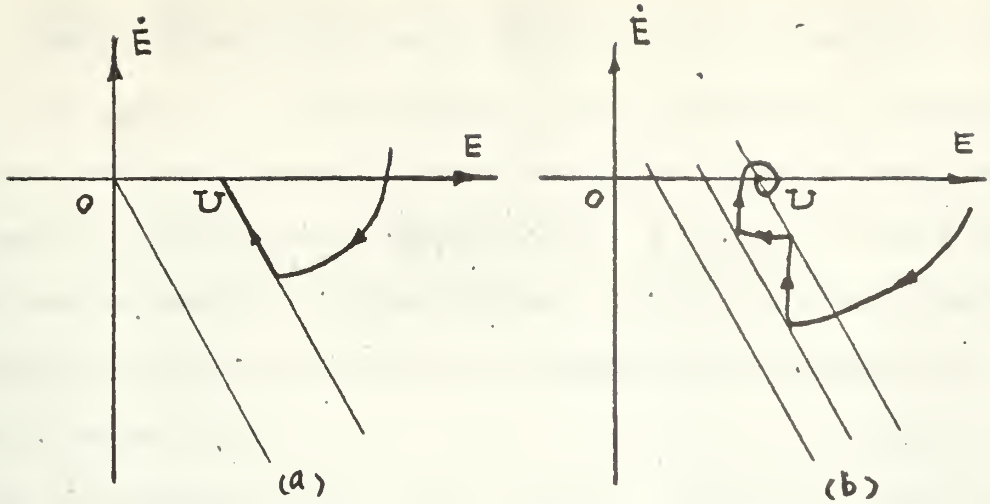


Fig. 5-8 Trajectories for Modified Switching
Line No. 1
(a) Ideal Relay
(b) Relay with Dead Zone

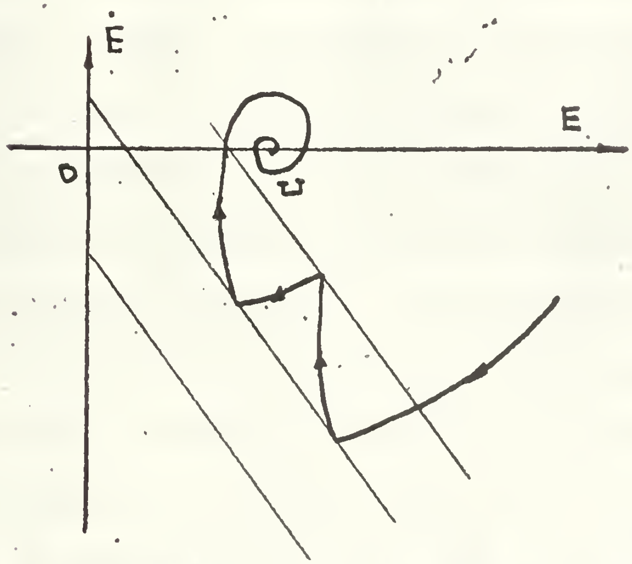


Fig. 5-9 Trajectory for Modified Switching
Line No. 1, with wide relay Dead
Zone.

Design of Switching Computer by Using Modified Switching Line #1B:

From Fig. 5-2, if the switching line is parallel to the fast eigenvector as for step input, and if the dead zone of the relay is negligible, then the signals for the switching computer is E, $\dot{E}$ and ω_i. E and $\dot{E}$ signals form the fast eigenvector for step response. ω_i translated the eigenvector in the right direction an amount due to the magnitude of the ramp input. The schematic diagram is in Fig. 5-10, where the relay R_3 is controlled by the $\dot{E}$ signal, opens the feedback circuit when the die-out trajectory of ramp response reaches the steady state point. If the dead zone can be neglected then the relays R_2 and R_3 can be taken off and the $\frac{a\omega_i}{K}$ signal fed into R_1.

III. Modified Switching line #2, by using the Switching Equation $E - \frac{a\omega_i}{K} = 0$:

From Fig. 5-2, for the heavily damped case the die-out trajectory for ramp input is nearly a straight line parallel to the vertical axis in the phase plane. The switching circuit can be designed by using E and ω_i as control signals. But in this case a switching operation is required to take out the damping as soon as the state point reaches the underdamped focal point. The trajectory is sketched in Fig. 5-11. The relay may operate at point p also; but since the direction of the trajectory is nearly the same for both the under and overdamped case, therefore, there will be no effect to the system.

The switching circuit is given in Fig. 5-12. The relay R_1 is the original relay circuit for step input, and the relays R_2, R_3 are for ramp input. R_2 forms the vertical switching line, and R_3 opens feedback as the $\dot{E}$ signal goes to zero.

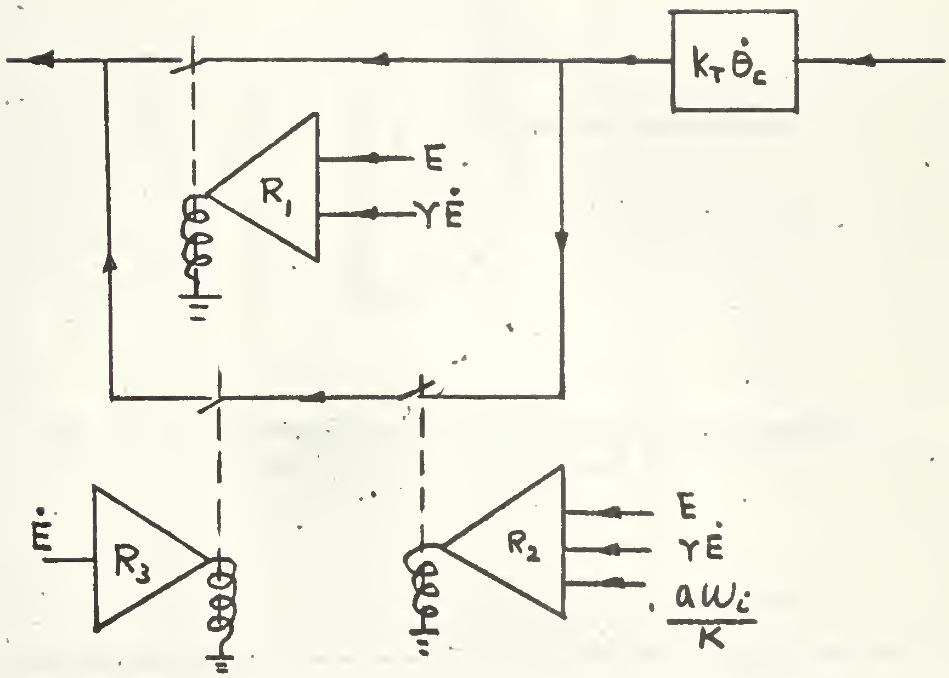


Fig. 5-10 Schematic Diagram of Switching Circuit for Using Modified Switching Line $\#1_b$.

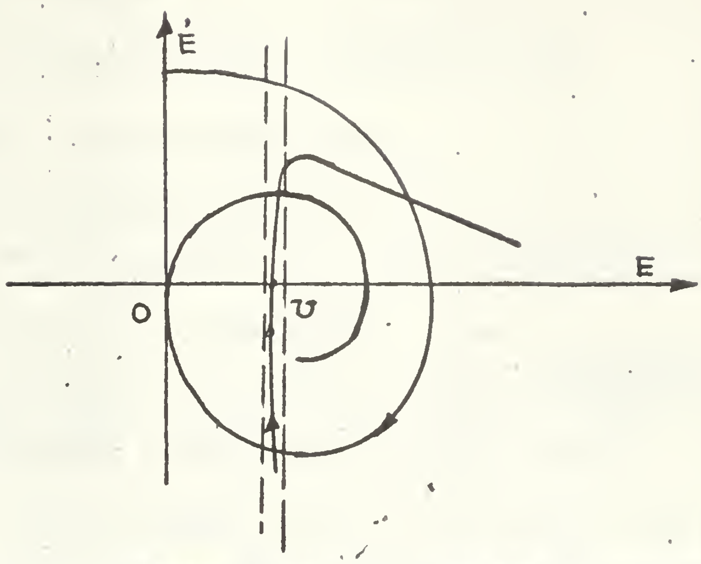


Fig. 5-11 Trajectories for Illustrating the Operation of Modified Switching Line #2.

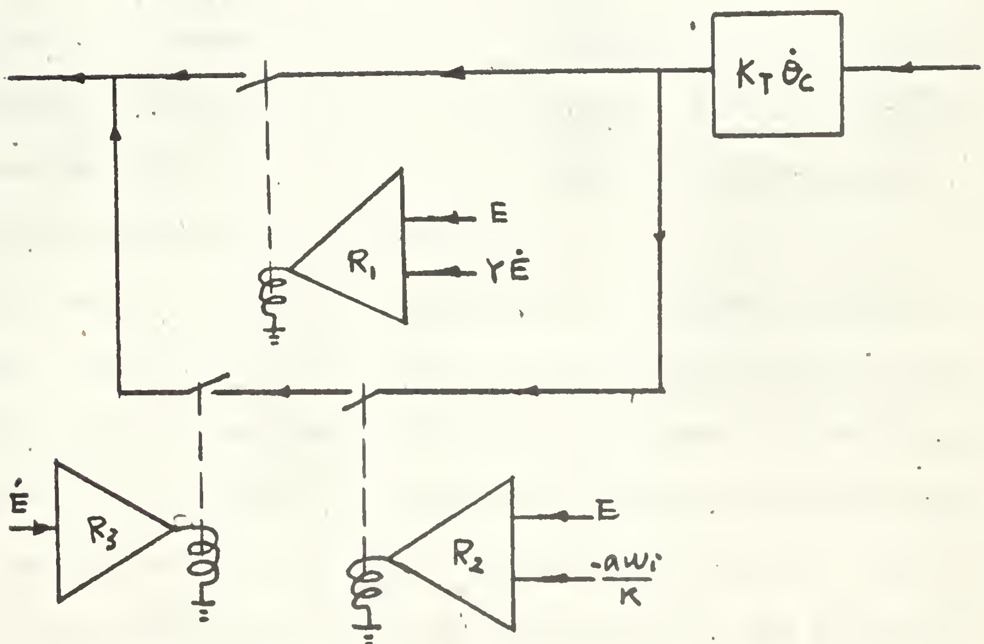


Fig. 5-12 Relay Circuit for Using Modified Switching Line # 2.

SECTION IV - TRANSLATED HYPERPLANE SWITCHING IN LINEAR ERROR SPACE FOR "THIRD ORDER AND HIGHER ORDER SYSTEMS WITH RAMP INPUT"

From second order system, the characteristic equation for ramp input is:

$$\ddot{E} + (a + KK_T)\dot{E} + KE = (a + KK_T)\omega_i \tag{5-6}$$

or

$$\ddot{E} + (a + KK_T)\dot{E} + K\left[E - \frac{(a+KK_T)\omega_i}{K}\right] = 0$$

Compare this equation to the step input case as

$$\ddot{E} + (a + KK_T)\dot{E} + KE = 0 \tag{5-7}$$

The only difference is the change of E to $\left[E - \frac{(a+KK_T)\omega_i}{K}\right]$

In the third order case, for the ramp input

$$\dddot{E} + (A + KK_a)\ddot{E} + (B + KK_T)\dot{E} - K\left[E - \frac{(a+KK_T)\omega_i}{K}\right] = 0 \tag{5-8}$$

$$\left[\begin{array}{l} \dddot{\theta}_c + A\ddot{\theta}_c + B\dot{\theta}_c = K(E - K_T\dot{\theta}_c - K_a\ddot{\theta}_c) \\ -\dddot{E} - A\ddot{E} + B(\dot{\theta}_R - \dot{E}) = K[E - K_T(\dot{\theta}_R - \dot{E}) - K_a\ddot{E}] \\ \text{Where} \quad \dddot{E} = -\dddot{\theta}_c, \quad \ddot{E} = -\ddot{\theta}_c, \quad \dot{E} = \dot{\theta}_R - \dot{\theta}_c = \omega_i - \dot{\theta}_c \end{array}\right]$$

Let $E' = E - \frac{B + KK_T\omega_i}{K}$ the equation becomes

$$\dddot{E} + (A + KK_a)\ddot{E} + (B + KK_T)\dot{E} + KE' = 0 \tag{5-9}$$

This is the form of the standard characteristic equation for step response. Then after translating the origin along the E axis by the amount $\frac{B+KK_T}{K}\omega_i$, all the trajectories within this shifted error space will be the same as that in original error space for step response.

Same as in Chapter II, three hyperplanes can be constructed for the overdamped case. But the switching computer is designed to switch so that the die-out trajectory chatters down and finally stays near the steady state point U , because at this point the trajectory inside the switching hyperplane tends to go out, but the trajectory outside tends to go in. The acceleration signal at this steady state point may not be zero, but the trajectory

cannot be far away from the steady state point in the projection on the E vs $\dot{E}$ plane, and the trajectory in this E vs $\dot{E}$ plane is the indicated actual value from the physical system. In other words the higher derivative signals are not being visualized as the velocity and error signals.

For a higher order system the translated hyperplane (space) switching theory can also be applied, because the only translation is in the E direction. The trajectory will oscillate and terminate at the steady state point of ramp input, the higher derivative signals may have considerable oscillation after switching occurs, but the projection of the trajectory on to the E vs $\dot{E}$ plane will still be a Z form in the dead zone of the relay.

<u>"Vertical Plane" switching for third Order and high order systems:</u>

From the phase plane plot of the second order system, the die-out trajectory due to a ramp input is nearly a straight line. Since the ramp input only translates the steady state point along the E axis in error space, then using a vertical plane for switching with a relay with moderate dead zone, the trajectory will be confined to the steady state point. But another relay must be instrumented to stop the trajectory at the steady state point as in the second order case.

The benefits of this switching method are : (1) - to eliminate the oscillation in the relay dead zone and (2) - it is easy to control the switching computer (only using E and $\dot{E}$ signals). But the stopping switch which operated at the steady state point may cuase some trouble, because the higher order derivative signals are apparently not zero here.

The step signal is an extreme case. It is even hard to produce a step signal unless by opening and closing a switch. Whenever a step signal passes through a R-C network or amplifier the sharp edge will be changed to exponential form. If a system only has the optimum character for step input, it will not be a good system in actual applications. But if the system can have optimum character for both step and ramp input, then such a system will give fast and accurate response both in theory and in practice.

The approximation of signals by using steps and ramps are illustrated in Fig. 5-13. The more segments used the better the result, and of course the best case is to use an infinite number of segments, then a continuous automatic adjustment in the control circuit will give best results.

The phase plane analysis of the second order system, with modified control computer for discontinuous damping, indicated that such a system can be optimized for step and ramp inputs at the same time, no matter how the input signal may change during the transient period, the computer will always throw in the damping feedback signal according to the optimized condition. In other words, the switching line is self adaptive to the input signal.

Note that the particular character of this switching computer is to use the feedback damping to optimize the system but does not introduce an additional lagging error for the ramp input as the usual continuous feedback compensated system does.

In this ramp response analysis the input signal analyzer supplies only the derivative signal of the input to the control computer. A type one system cannot be expected to follow an input signal with higher derivatives. But the idea of "Translated error space" can give a clear view about how the trajectory changes in the phase space due to input signals other than a step

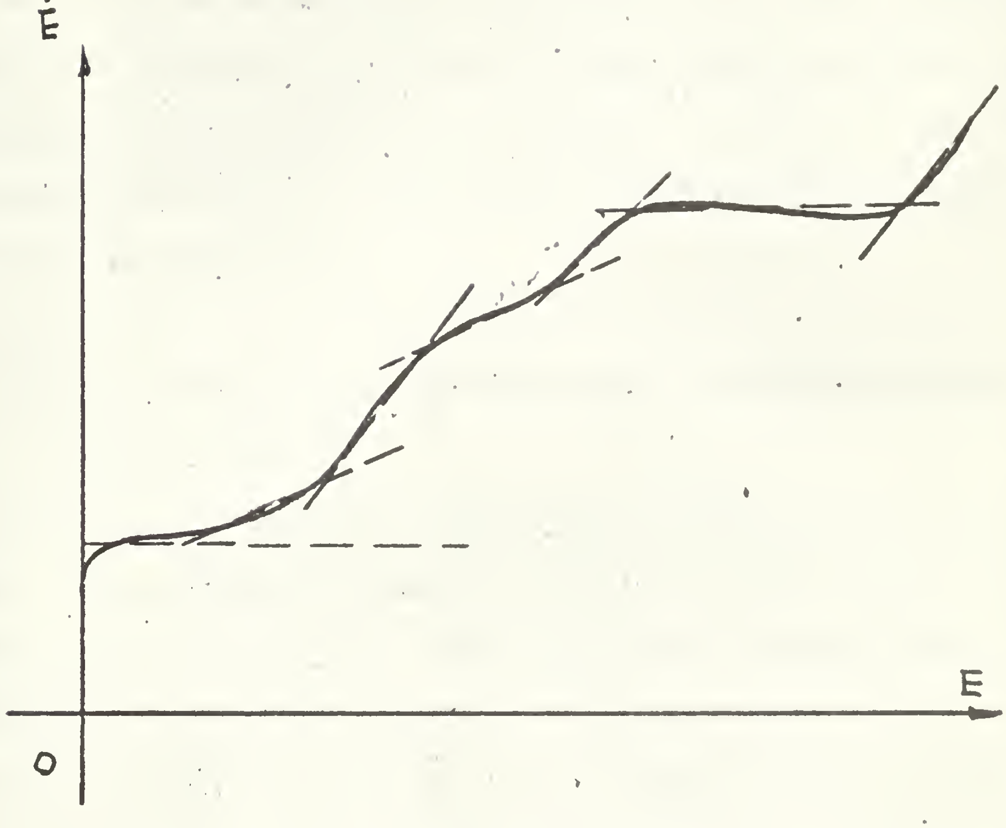


Fig. 5-13 Step and Ramp Approximation to a Complex Input Signal

input. Whether the same approach can be used to analyze type two or type three systems is an important thing to be investigated. In order to make the control computer provide the system with optimum response for all kinds of input, then the character of the input signal analyzer is an essential factor. In Chapter I the description and definition of phase space indicated that the futures of the variable in phase space, as the time increases are decided by both the initial conditions and the forcing function. This means the forcing function and initial conditions are both equally important. This study has used the error space for step input (we may call this a stationary error space). It is helpful to consider the forcing function as only a change in the coordinates of this stationary error space. This may be called a moving or translating error space, because the axes used are still the error signal and its time derivatives.

SECTION VI - ANALOG COMPUTER STUDY FOR SECOND ORDER DISCONTINUOUS SYSTEM WITH RAMP INPUT.

The second order discontinuous system used in Chapter II with h = 0.3 is used here. The equations for computer setup are the same, except that the control signals are E $\dot{E}$ and ω_i. The circuit of the computer setup is given in Fig. 5-14, and the setting values are in Table 5-1. The ramp response curves for different magnitudes of input signals by using an unmodified switching line are recorded in Fig. 5-15A. The die-out trajectories are nearly vertical lines and there are oscillations around the underdamped focal points. All these agree with the analysis given before. The next thing to do is to use various modified switching lines to check the results of the phase plane analysis given before. One recording as an example is given in Fig. 5-15b, where the translated eigenvector is used as the switching line, the drop out lines of the relay are shifted in the positive error direction

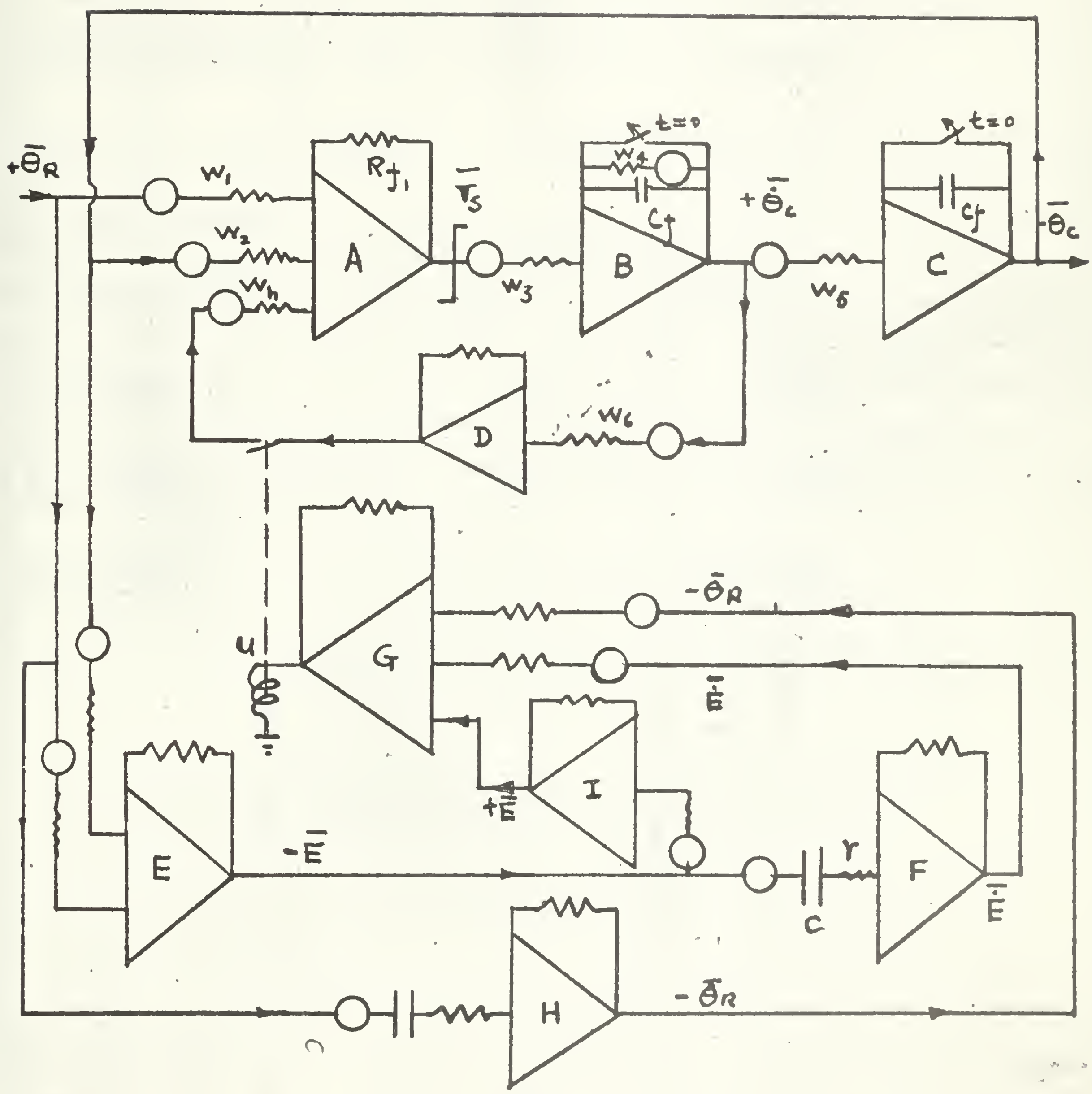


Fig. 5-14 Computer Setup for Discontinuous Second Order Saturated System with Ramp Input.

TABLE 5-1 Setting Values for Analog Computer Setup in Fig. 5-14.

$W_E = 1$		$R_{fE} = 1$	$R_{e1} = 0.5$	$a_{e1} =$	0.5
			$R_{e2} = 0.5$	$a_{e2} =$	0.5
$W_I = 1$		$R_{fI} = 1$	$R_I = 0.5$	$a_I =$	0.5
$W_F = 1$	$C = 0.0025$	$R_F = 10$	$Y = 0.1$	$a = 0.4$	
$W_H = 1$	$C = 0.25$	$R_H = 10$	$Y = 0.1$	$a = 0.4$	

$\alpha_G = \alpha\theta_c = 0.00996$

$\alpha w_i = \alpha\dot{\theta}_c = 0.0704$

$\alpha_u = 1$

$$u = -\frac{\gamma_a}{K} w_i + \gamma E + \dot{E} = 0$$

$$\bar{u} = -W_{G_1}\bar{w_i} + W_{G_2}\bar{E} + W_{G_3}\bar{\dot{E}}/\omega_n$$

$W_G = 0.653$		$R_1 = 1$	$a_1 = 0.653$
$W_{G2} = 2.288$	$R_G = 1$	$R_2 = 0.1$	$a_2 = 0.2288$
$W_{G3} = 0.88$		$R_3 = 0.1$	$a_3 = 0.704$

R in $M\Omega$, C in μf

The other setting values are the same as in Fig. 2-4.

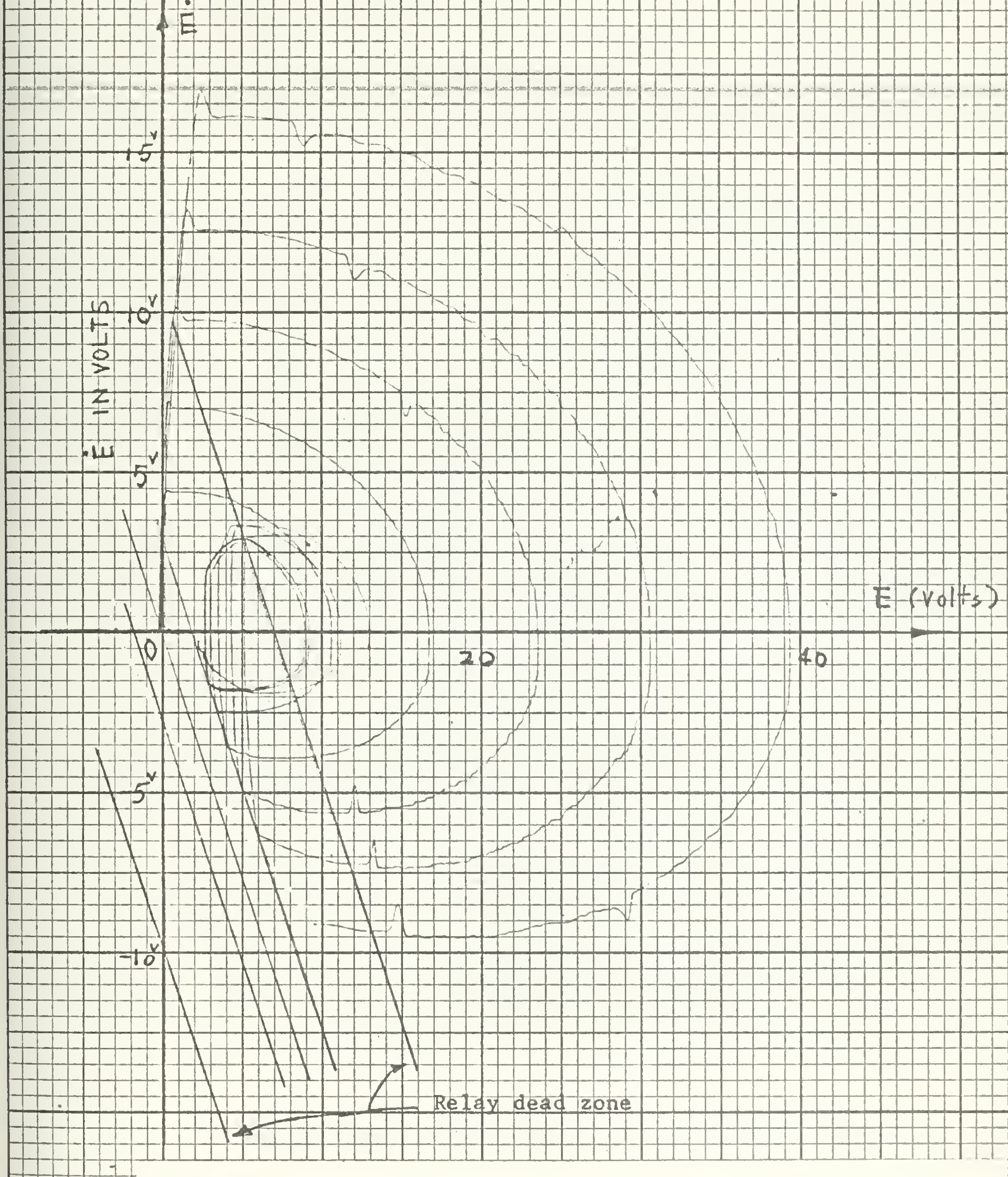


Fig. 5-15a Ramp Response Curves of Second Order Discontinuous System with Unmodified Switching Line.

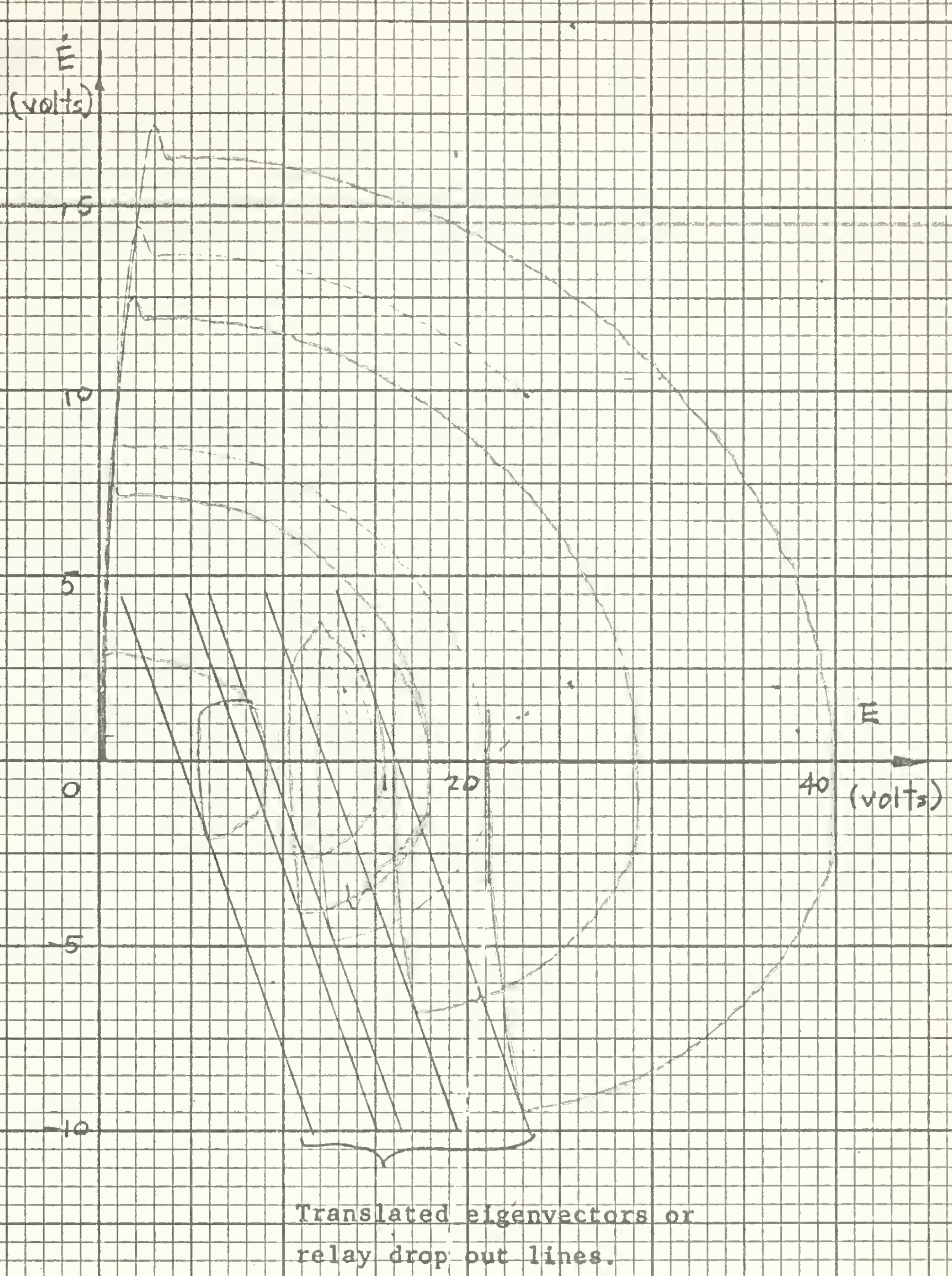


Fig. 5-15b Ramp Response Trajectories of Second Order Discontinuous System with Translated Eigenvector as Switching Line.

according to the magnitudes of ramp inputs the oscillations near the under-damped focal points are reduced, but since a wide relay dead zone is used here, therefore there is no chattering but there are some velocity overshoots as indicated in the figure. The various computer results using narrower relay dead zone and different switching lines are recorded in Fig. 5-15c to Fig. 5-15g.

SECTION VII- RAMP RESPONSE ANALYSIS FOR SECOND ORDER SATURATED SYSTEMS.

In the recently developed theory of relay servos or bang-bang systems usually the designer considers a saturated system with high gain in the main amplifier as belonging to the relay servo type. Actually this is one case of a discontinuous system. When the main amplifier is saturated, the feedback damping loses its effect. When the trajectories come out from the saturated region the damping effect returns and makes the damped trajectories have better character. Also the feedback damping signals act as switching signals at the same time.

As indicated in Chapter III and IV, because of the saturation effect of the main amplifier, the system working in the linear region is always under the critical or overdamped conditions. It however, belongs to the class of fully saturated systems or semilinear systems. Since the small step input is to be neglected in the original assumption of the conventional analysis, then the system is the same as a relay servo-system. But one thing that cannot be neglected is the response character of the ramp input. Tachometer feedback in over damped systems will cause the lagging error due to the ramp input to be increased. Even if the specification of the system has no limitation on the ramp response character, but the trajectory of the system due to a ramp input works in a linear mode or linear and saturated combined

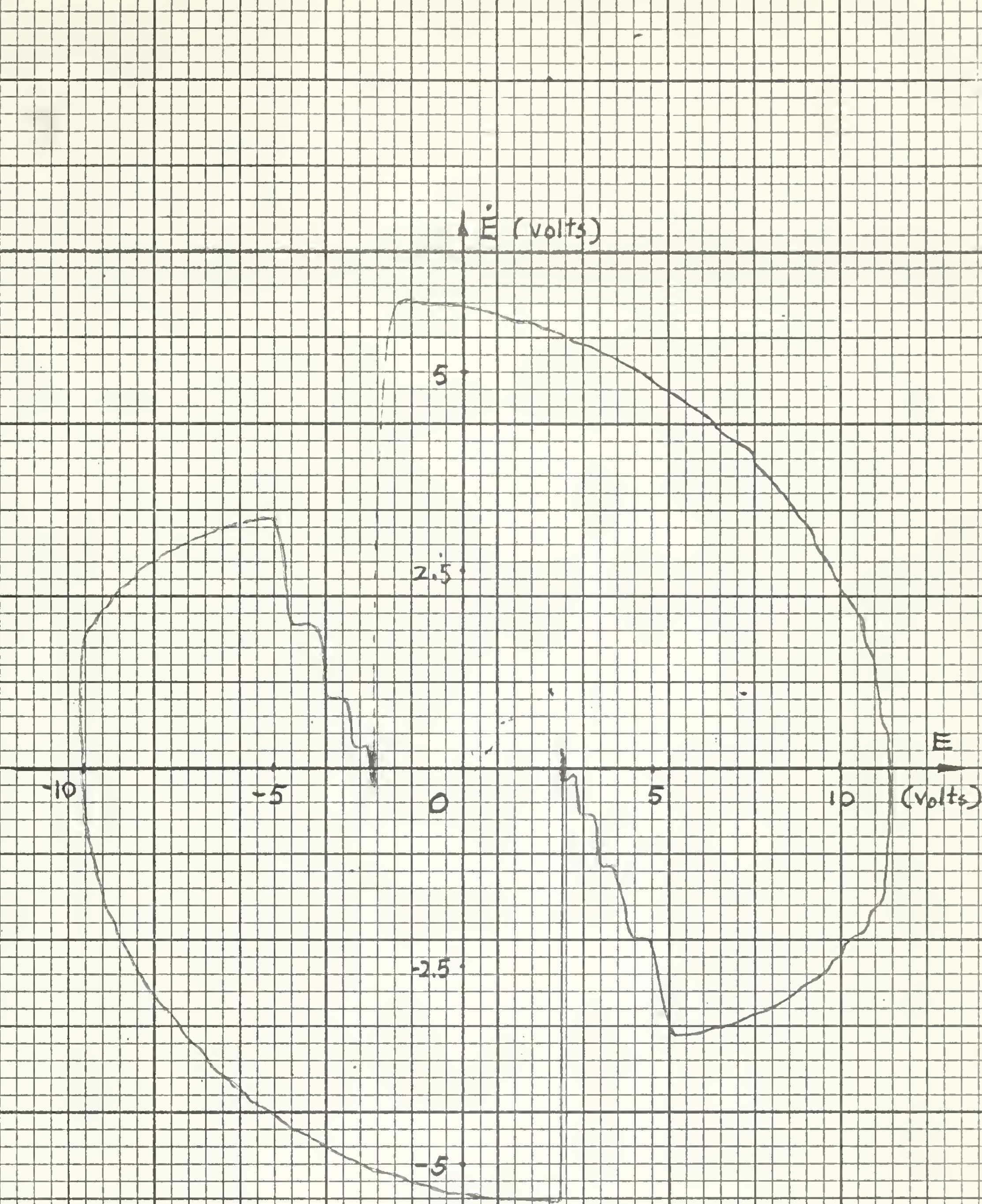


Fig. 5-15c Ramp Response plot for a 2nd order system with Discontinuous Feed back and Translated Eigenvector Switching (Computer result).

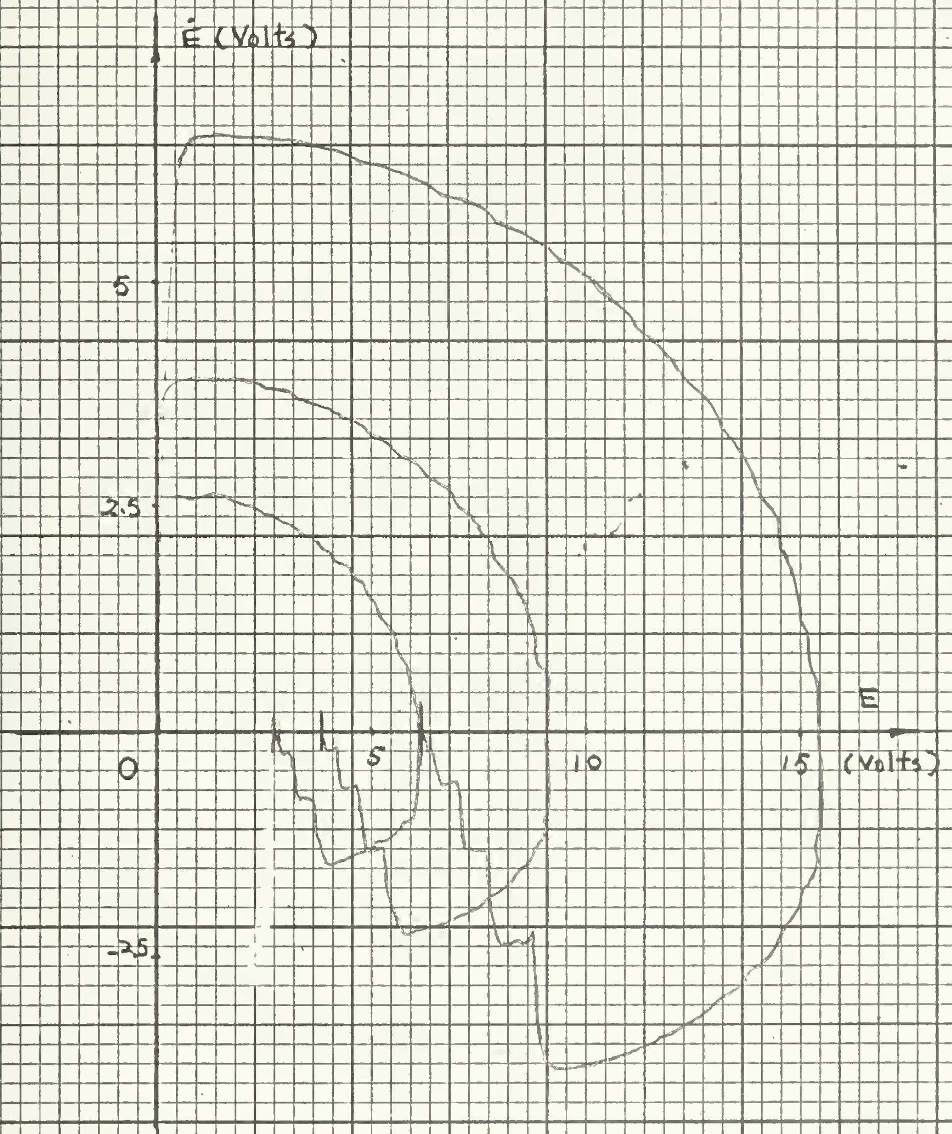


Fig. 5-15d Ramp Response Plots for Different Magnitude of Ramp Inputs.

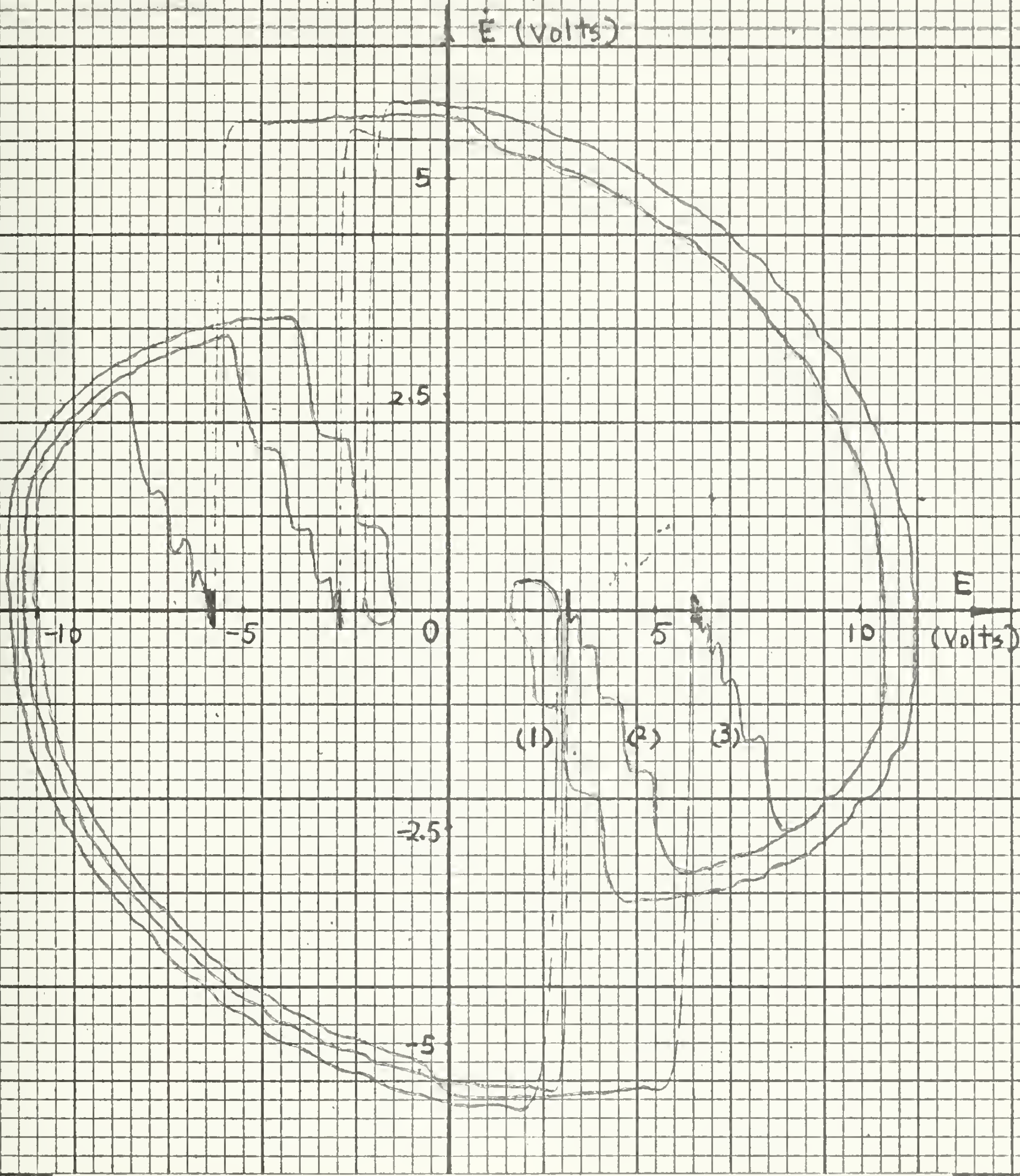


Fig. 5-15e Ramp Response Plots for using Different Magnitude of Modification Signal $\frac{a\omega_c}{K}$

(1) Switched too late.
(2) & (3) Switched too early.

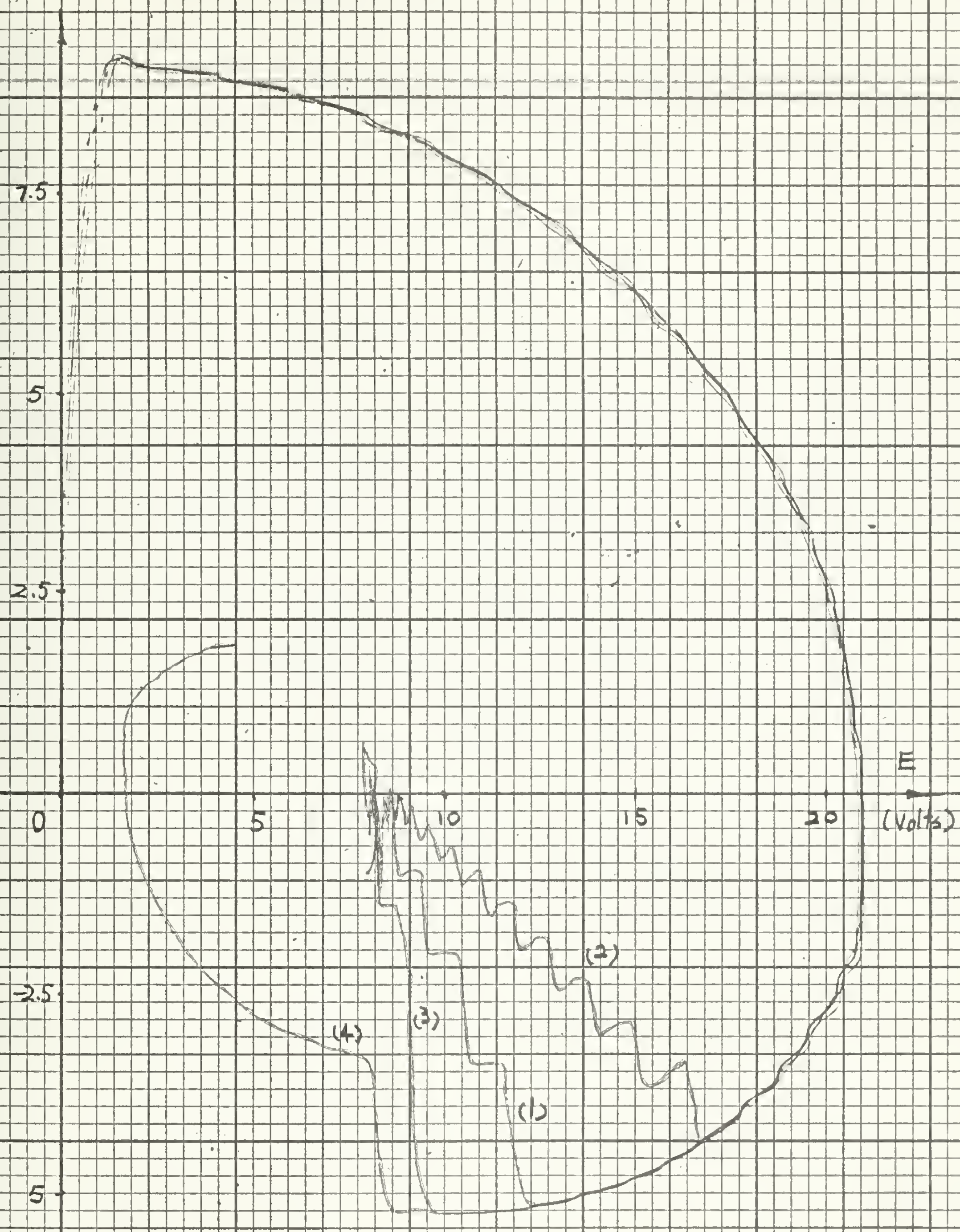


Fig. 5-15j Ramp Response Plots by Using Switching Lines with Different Slopes

(1) Translated Eigenvector Switching
(2), (3) & (4) Switching Lines with Various Slopes.

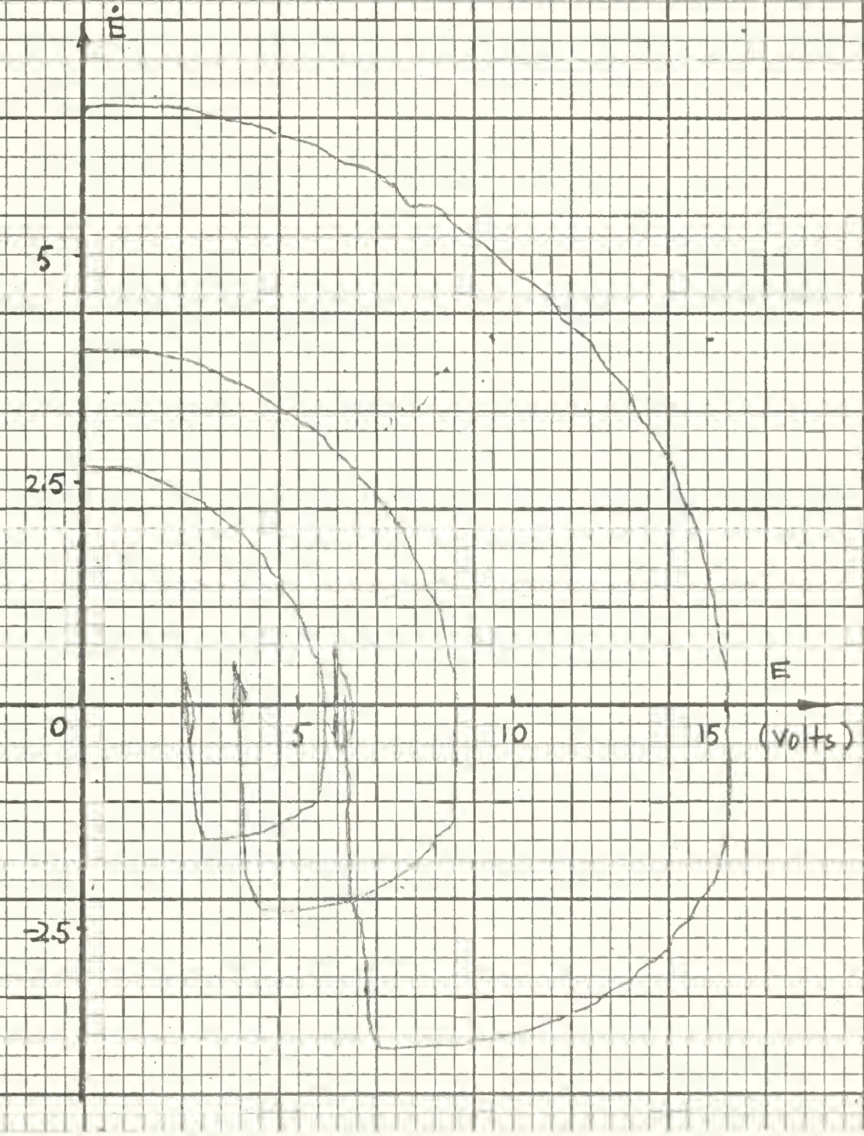


Fig. 5-15g Ramp Response Plots for Different Magnitude of Ramp Inputs, with nearly Vertical Switching Line.

mode, it is apparent that the theory of a relay servo cannot apply to this situation very well. This statement is classified in the later analyses.

From Chapter III the effect of tachometer feedback is to change the slope of the linear zone. In this chapter phase plane analysis is used to indicate the effect of tachometer feedback also. The lagging error and the method for reducing it will be discussed based on the theory of discontinuous systems. The general block diagram is in Fig. 5-16.

The general characteristic equation for linear second order systems with ramp input as used before is:

$$\ddot{E} + (a+KK_T)\dot{E} + KE = (a+KK_T)\omega_i \tag{5-11}$$

The isocline equation is:

$$\frac{E}{\omega_n} = \frac{(K_T + \frac{a}{\omega_n^2})\omega_i - E}{N + \frac{KK_T + a}{\omega_n}} \qquad \text{where} \quad N = \frac{d\dot{E}/\omega_n}{dE} \tag{5-12}$$

The lagging error is:

$$\varepsilon(t)\Big|_{t=\infty} = \frac{a\omega_i}{K} + K_T\omega_i \tag{5-13}$$

The general equation for saturated second order systems with ramp input is

$$\ddot{\theta}_c + a\dot{\theta}_c = \mp V_S$$

Since $\ddot{\theta}_c = -\ddot{E}$ $\quad \dot{\theta}_c = \dot{\theta}_R - \dot{E} = \omega_i - \dot{E}$

Then

$$\ddot{E} + a\dot{E} = -V_S + a\omega_i \tag{5-14}$$

The equation of isoclines is:

$$\frac{\dot{E}}{\omega_n} = \frac{a\omega_i - V_S}{\omega_n^2\,(N + \frac{a}{\omega_n})} \tag{5-15}$$

For linear operation the open loop transfer function is

$$F_o(S) = \frac{K}{S^2 + (a + KK_T)S} \tag{5-16}$$

$$E(S) = \frac{S^2 + (a - KK_T)S}{S^2 + (a + KK_T)S + K}\,\bar{\theta}_i = \frac{[S^2 + (a + KK_T)S]\,\omega_i}{[S^2 + (a + KK_T)S + K]\,S^2} \tag{5-17}$$

The lagging error is

$$E(S)\Big|_{t=\infty} = \lim_{S\to 0} S\,E(S) = \frac{a\omega_i}{K} + K_T\omega_i \tag{5-18}$$

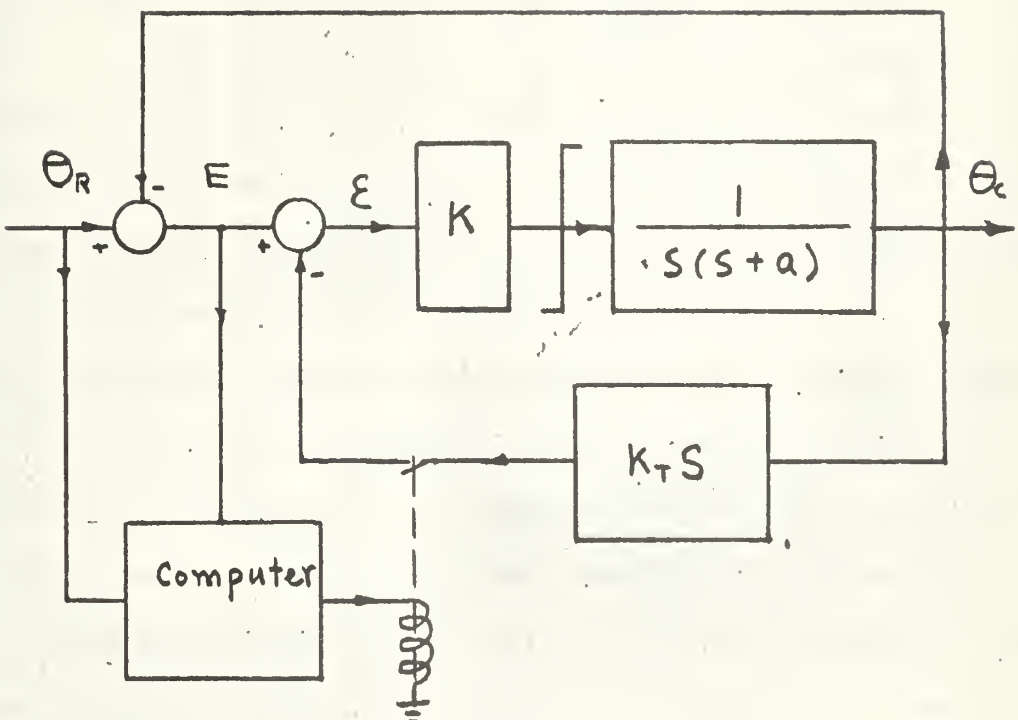


Fig. 5-16 General Block Diagram of Second Order Saturated System with Discontinuous Tachometer Feedback.

The slope of the linear zone is decided by the equation $E - K_T\dot{\theta}_c = 0$

i.e. $$E - k_T(\dot{\theta}_R - \dot{E}) \doteq 0$$

or $$E + K_T \dot{E} - k_T\omega_i = 0 \tag{5-19}$$

when $\dot{E} = 0$, then $E = K_T\omega_i$

because the gain of such a system is usually very high, the lagging error $E(s)\Big|_{t=\infty}$ stays in the linear zone or near the center line of the linear zone (for a system with large and small K this is not true).

By assuming numerical values for the equations given above (K = 1000, a = 2.6, V_s = 10, K_t = 0.0604, ω_i = 0.1, 0.236, and 1) the phase plane plot is as in Fig. 5-17, 18, 19. All of them are purely linear trajectories. The lagging error is about six percent. The reason for linear operation is due to the tachometer feedback which turns the linear zone from the vertical to the positions indicated in each figure.

For a larger ramp input the main amplifier will be saturated, the general shape of the trajectory is as in Fig. 5-20. The isocline equation in the saturated region is different for every magnitude of ramp input even if the output of the saturated amplifier is kept constant. If the system has a large value of a and small value of V_S, then the saturated trajectory can not curve back as in Fig. 5-20. The response of the system for a ramp input will be very slow. But when the value of a is small, and V_S is large, then the system in the saturated region is underdamped, the trajectories in the saturated region tend to come into the linear zone rather than leaving it. From equation (5-15) the gain of the main amplifier still has effect upon the saturated ramp response trajectory, but not like that in Chapter III where the saturated trajectory for a step input is independent to the magnitude of input and the main amplifier gain.

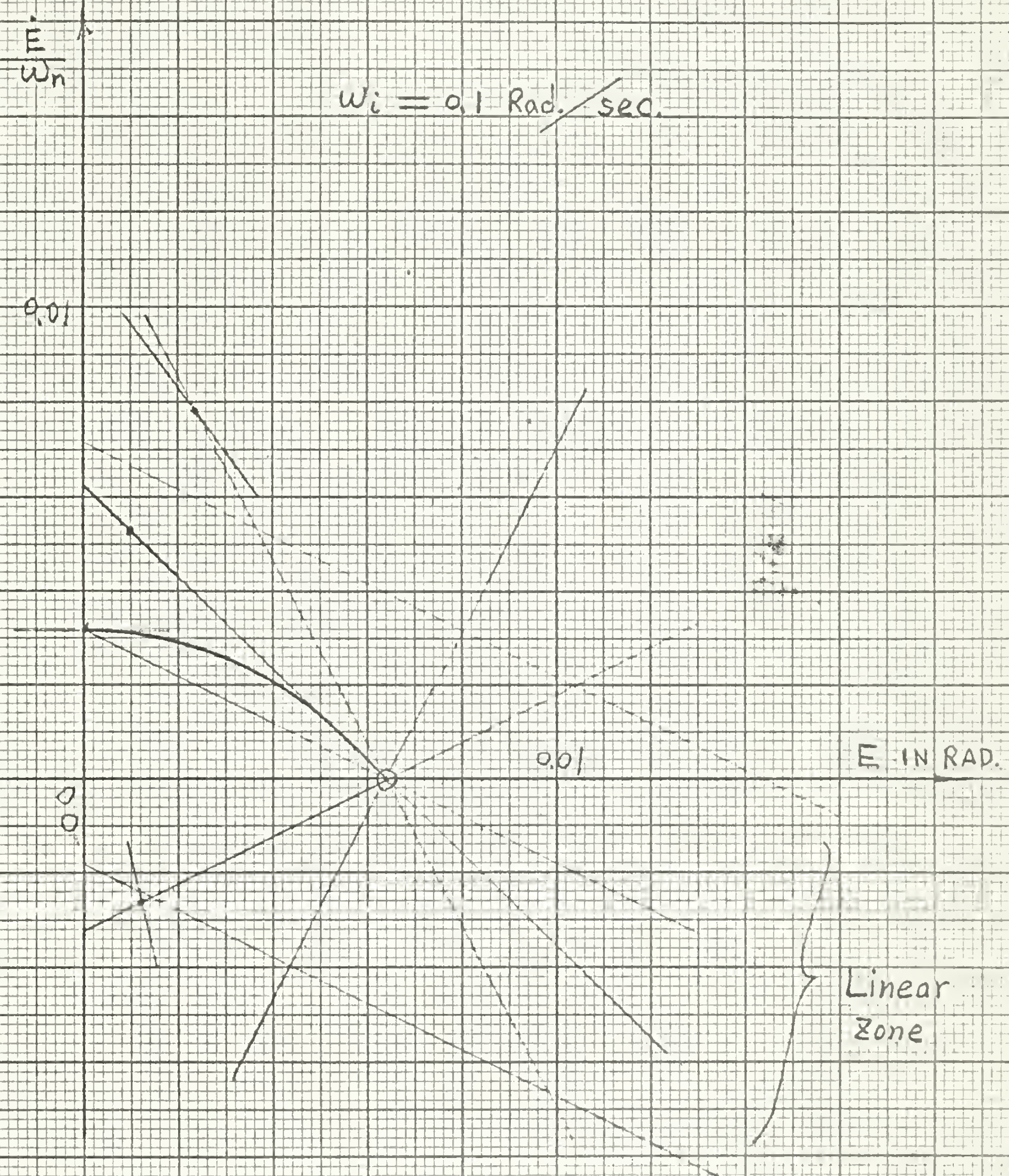


Fig. 5-17 Ramp Response Trajectory for Saturated Second Order System with ω_i = 0.1 rad/sec.

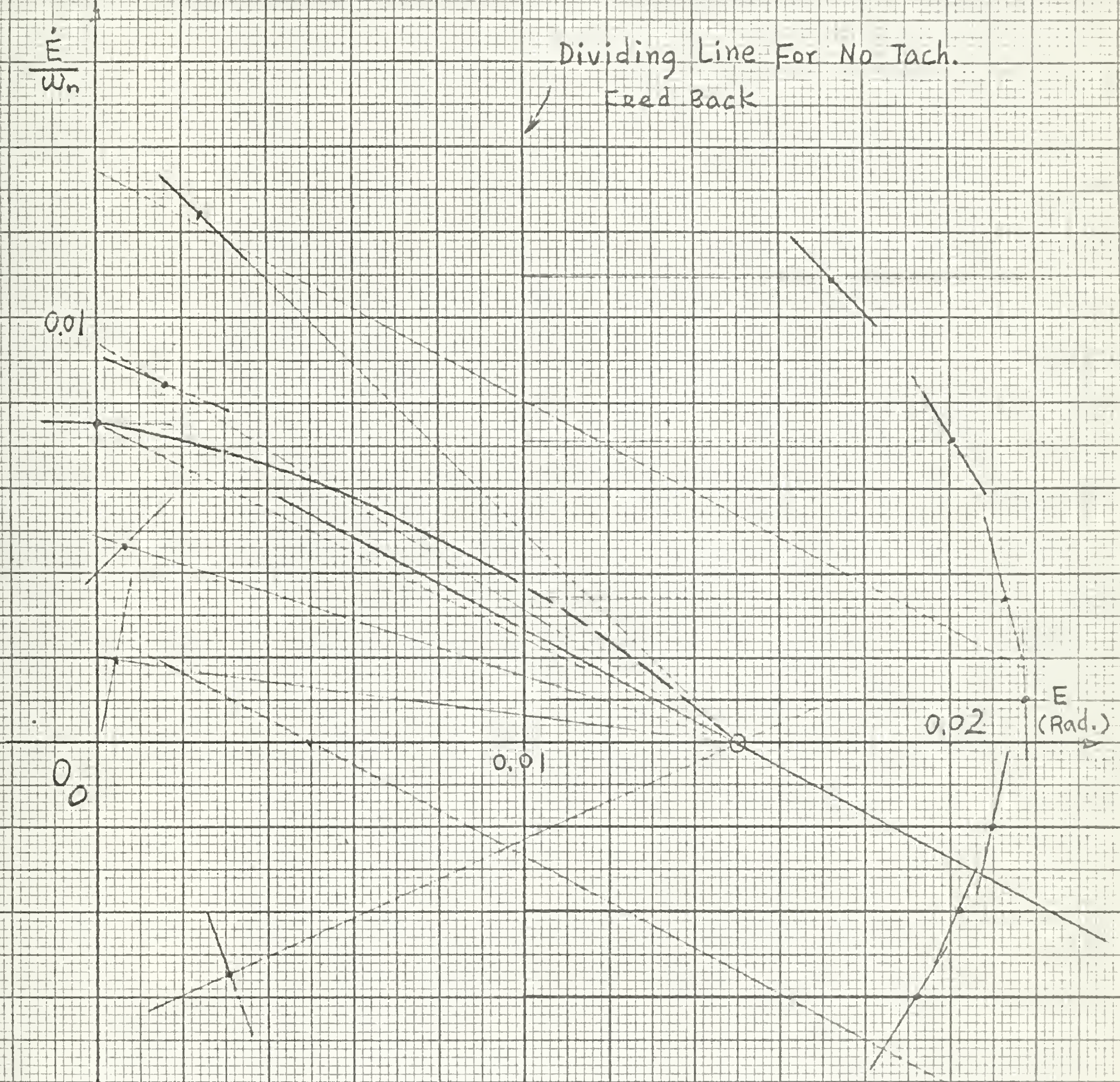


Fig. 5-18 Ramp Response Trajectory for Saturated Second Order System with ω_i = 0.236 rad/sec.

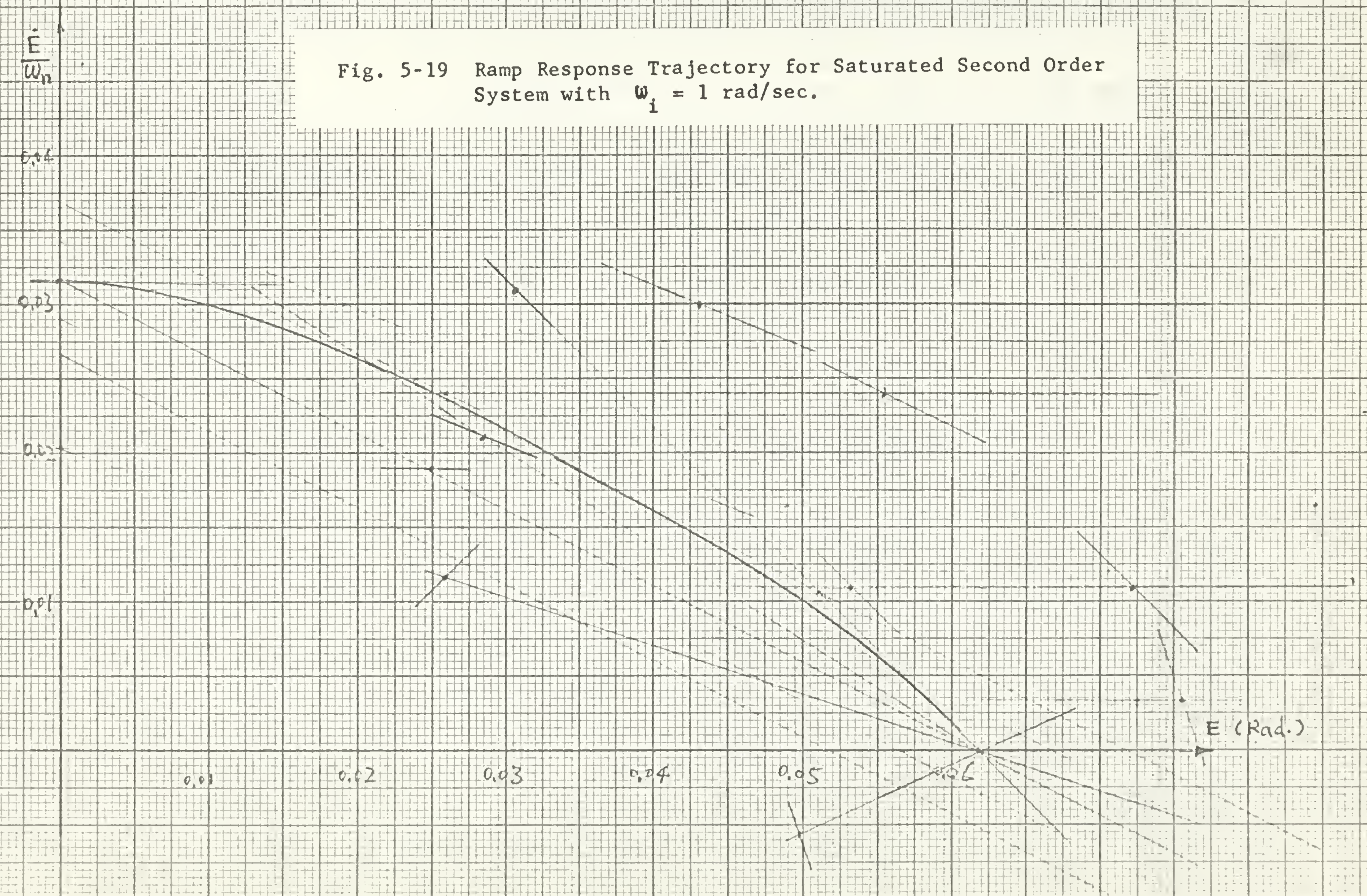


Fig. 5-19 Ramp Response Trajectory for Saturated Second Order System with ω_i = 1 rad/sec.

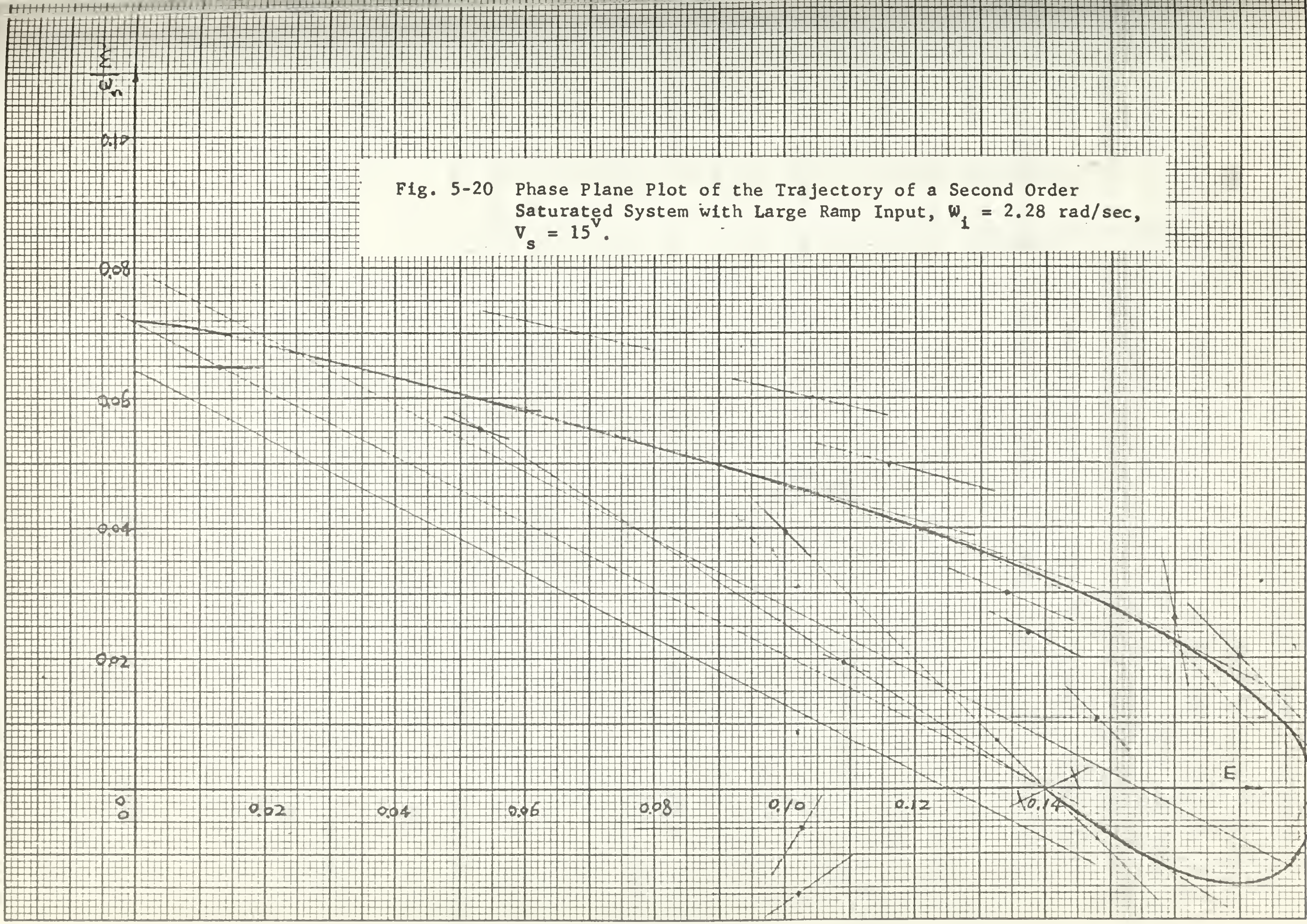


Fig. 5-20 Phase Plane Plot of the Trajectory of a Second Order Saturated System with Large Ramp Input, $W_i = 2.28$ rad/sec, $V_s = 15^V$.

Since the underdamped trajectory for positive ramp input will turn back to the negative $\dot{E}$ axis, then it is possible to operate the system with no feedback at the first part of the trajectory, the trajectory will be saturated and spiral around the origin of the phase plane, and by switching in the feedback at a proper instant the die-out trajectory will pass through the stable point of the underdamped condition, and no additional lagging error will be caused by the large tachometer feedback, provided that the feedback signal can be switched off at that point.

By assuming $\omega_i = 2.28$, $V_s = 25$, the other values as before, then the various trajectories are plotted in Fig. 5-21. In this case the lagging error is 0.14 rad. for a saturated system with no switching action. But by using discontinuous feedback the lagging error can be reduced to 0.00594 radian. Since the die-out trajectory is not a vertical line, then the translated eigenvector switching method will give better results. (Here only one eigenvector, because the system is under critically damped conditions). Before the feedback signal is switched in, the trajectory is moving in the downward direction; after the switch closes, the trajectory tends to go upward, then the switching line is operated in a converged region. It may have some chattering but the trajectory will stay in the dead zone of the relay, go to the underdamped stable point and stop there, because the overdamped trajectory will make the relay open, and the underdamped trajectory will make the state point stop there.

The analog computer study for a critically damped saturated system with K = 1000, a = 2.6 is given in Fig. 5-22, the setting values are in Table 5-2. The recorded plots are in Fig. 5-23, they are checked with the phase plane plots.

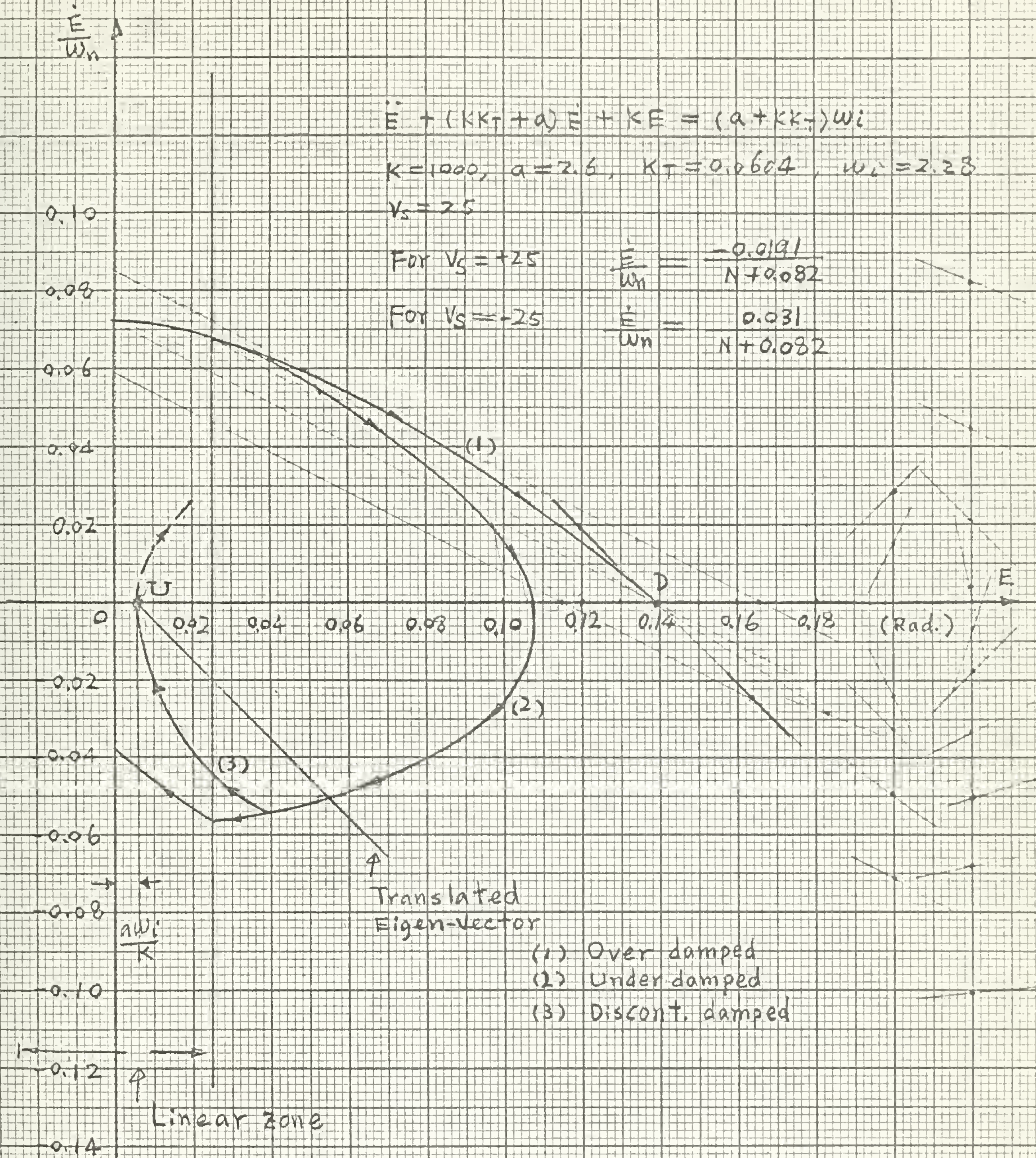


Fig. 5-21 Phase Plane Plot for Illustrating the Various Trajectories and Switching Lines for a Discontinuous Second Order Saturated System.

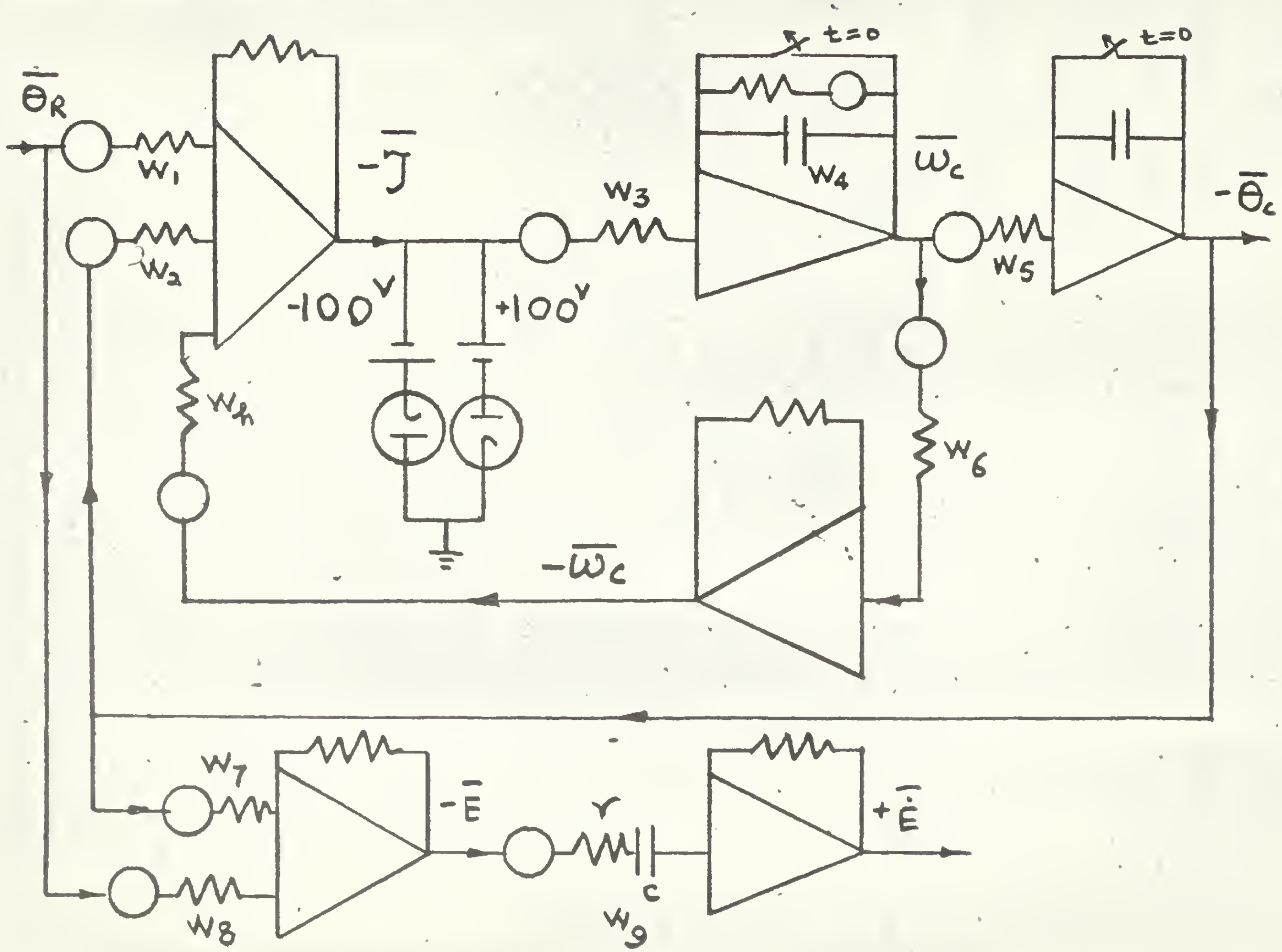


Fig. 5-22 Computer Setup for a Second Order Saturated System.

TABLE 5-2 Setting Values for Computer Set up for a Second Order Saturated System with K = 1000.
a = 2.6, H = 0.06064 (critically damped).

Scaling:

α_t =	20
α_{θ_R} =	0.01
α_{θ_C} =	0.01
α_y =	0.1
α_ω =	0.0158

Setting values:

$W_1 = W_2$=	100	R_{f1} =	10	$R_1 = R_2$	= 0.1	$a_1 = a_2$	= 1
W_h=	9.59	"	"	R_h	= 0.1	a_h	= 0.0959
W_3=	0.316	C_{f3} =	1.06	R_3	= 1.0	a_3	= 0.335
W_4=	0.13	"	"	R_4	= 1.0	a_4	= 0.138
W_5=	0.079	C_{f5} =	1.05	R_5	= 1.0	a_5	= 0.083
W_6=	1	R_{f6} =	1.0	R_6	= 0.5	a_6	= 0.5
W_7=	1	R_{f7}=	1.0	R_7	= 0.5	a_7	= 0.5
W_8=	1	"	"	R_8	= 0.5	a_8	= 0.5
W_9=	1	R_{f9}=	10	C	= 0.25	a_9	= 0.4

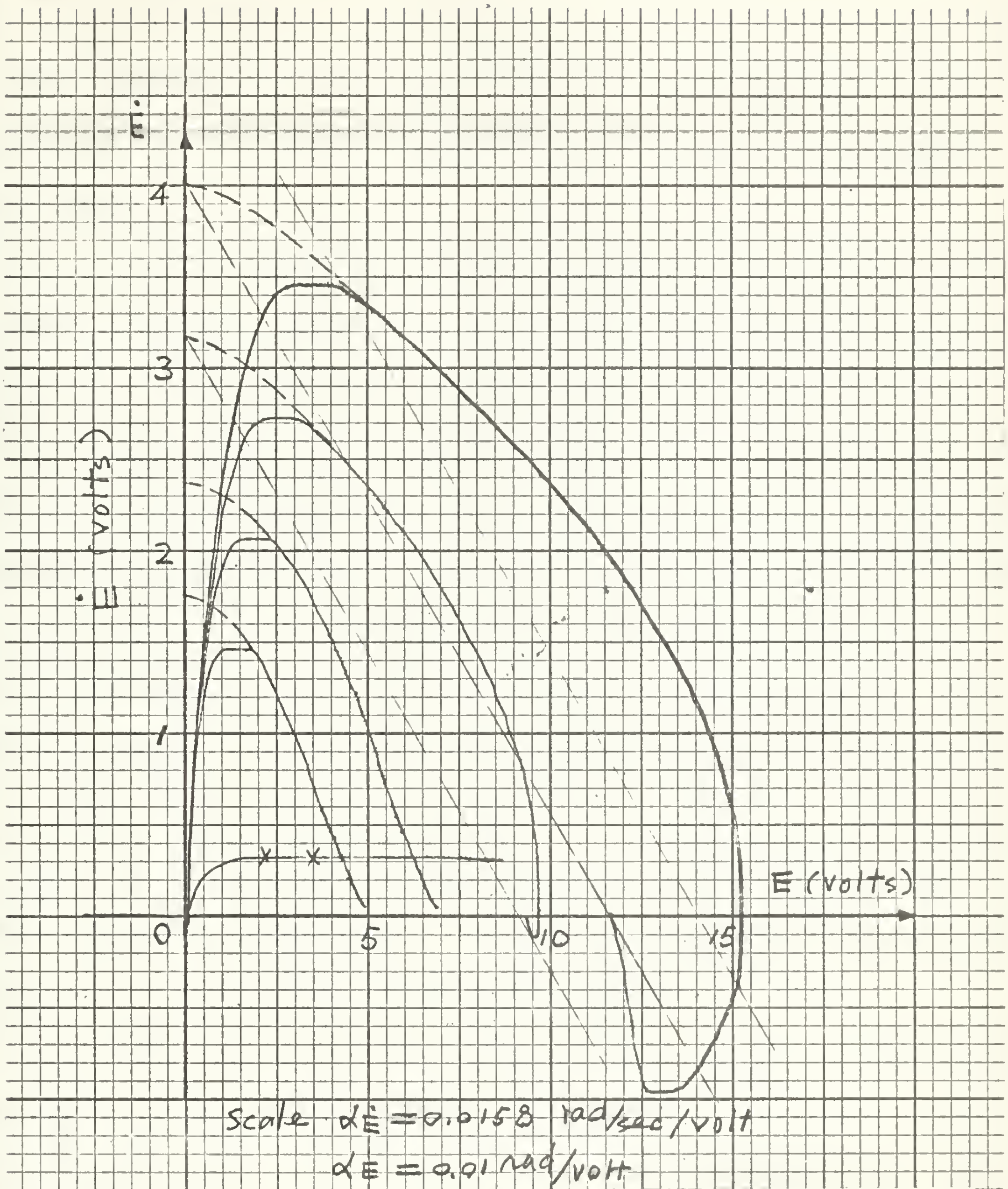


Fig. 5-23 Ramp Response of Saturated Second Order System. (Computer Result.)

SECTION VIII - DISCUSSION ABOUT RAMP RESPONSE FOR SEMILINEAR AND SATURATED HIGH ORDER SYSTEMS BY USING THE PHASE SPACE CONCEPT.

By using the theoretical background from the previous Chapters and the ramp response analysis for the linear systems, it is possible to predict how the trajectory of a third order semilinear or saturated system traverses the error space.

From the starting point and the final lagging error on the E axis the axisal plane is moved in to the 1 and 1' quadrant and the intersections with positive E and $\dot{E}$ axes are decided by the magnitude of the ramp input and the final lagging error corresponding to the overdamped case. The trajectory for the underdamped case will spiral in the space according to the sequence of the quadrant 1', 4', 4, 3, 2, 2', 1', 4' (See Fig. 1-7). The trajectory corresponding to the overdamped case will only move in quadrant 1'. (All these discussions are based upon a positive input ramp signal). The overdamped trajectory and the final part of the underdamped trajectory are nearly in planes, the intersection of these planes (the intersection with the E vs $\dot{E}$ plane) are far from the $\dot{E}$ axis or near the $\dot{E}$ axis as determined by the overdamped characters respectively. For a small ramp input there will be no saturation both in semilinear and saturated systems.

In semilinear systems, because the coefficients in the characteristic equation are large (See Chapter IV) then the saturated trajectory is usually in the overdamped form. In this case it may not be possible to use a discontinuous switching method, because the trajectory will never turn back to the $\dot{E}$ vs $\ddot{E}$ plane. But this is not always true, especially when the amplifier gain is high and the saturated output of the amplifier is large.

On the other hand, in saturated systems, because the coefficients in the characteristic equation are small, it is very easy to have an underdamped

character after the application of a ramp input large enough to cause saturation. Then the translated hyperplane switching is applicable. The converged character will be the same as in second order saturated systems or third order linear systems with ramp input.

For higher order systems, a clear view is not available using the phase plane or phase space method, but one thing is in common for all cases "for a system under saturated (with no feedback) condition if it is underdamped, then the translated hypersurface switching method can be applied to eliminate the additional lagging error caused by a ramp input.

Suggested for further investigations:

1. Translated error space theory applied to higher type systems.
2. Computer study using various modified switching methods.

CHAPTER VI - ANALYSIS AND DESIGN USING DAMPING TECHNIQUES FOR VARIOUS DISCONTINUOUS SYSTEMS

GENERAL DESCRIPTION

In the previous chapters, the general assumption made for the discontinuous system is that the character of the system can be changed from underdamped to overdamped by a simple action (Relay, computer, or other device). The general theory about how to change the damping condition at a proper instant according to various input signals has been discussed, the design problems and some of the methods of approximation have been presented. But the general analysis and design showing how to produce these underdamped and overdamped characters haven't been given. Since the characters of each kind of system is different from another, in some cases it is not possible to design a system that has these two kinds of characters, then the theory of discontinuous system is not applicable to all the control systems unless they meet the general assumptions as mentioned before.

In this chapter, the theory and the design problems about producing underdamped and overdamped characters for the various control systems will be discussed extensively. The root locus method and the frequency response method both will be used to determine the damping conditions of the system. The general approach is based on the approximation theory, there is no complex computer, and even the essential compensating signals may not be available. Since the modification of the basic theory of switching is different, then a specific discussion about the design of computers will be given whenever necessary.

For convenience the general block diagram for discontinuous systems in Chapter I is redrawn here in Fig. 6-1.

The analysis may concentrate on one of the blocks, or consider a couple

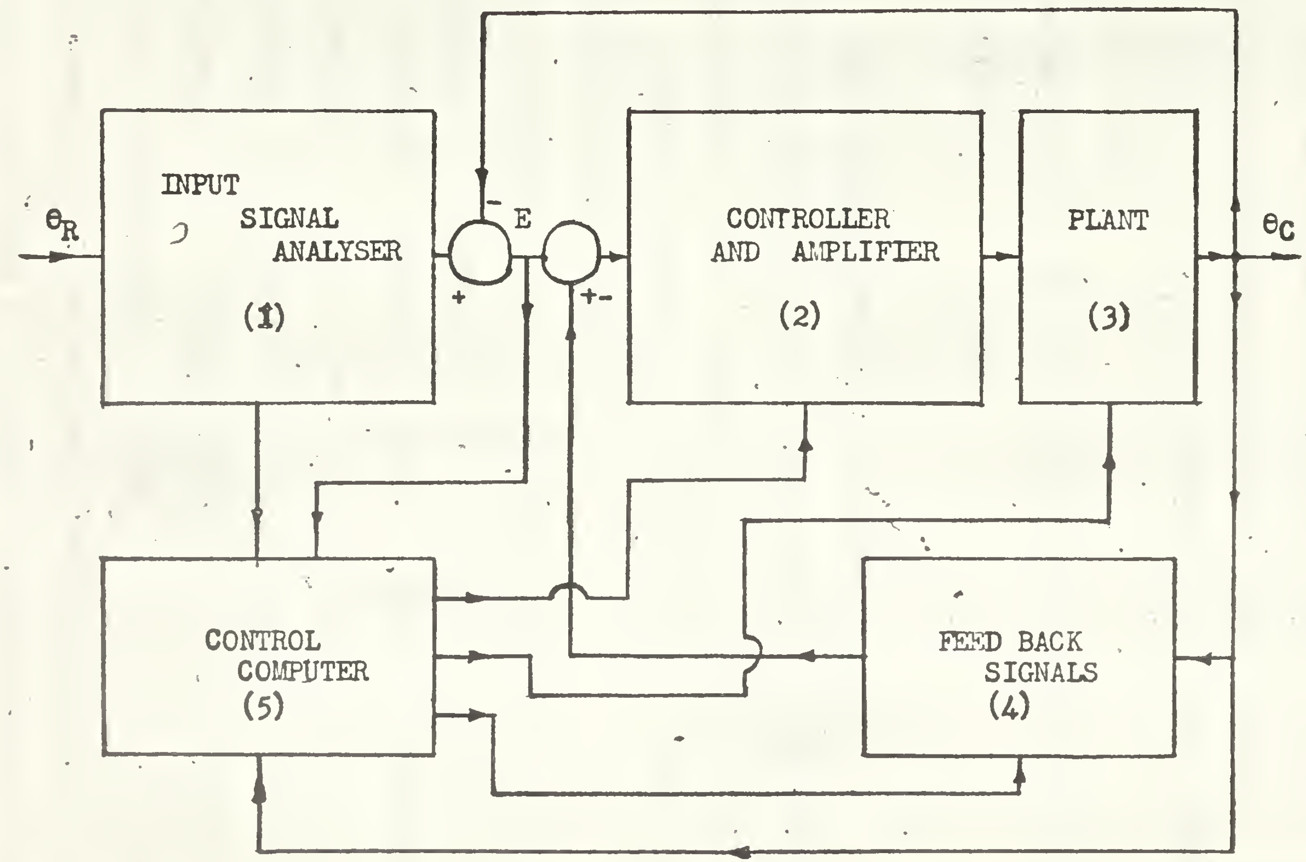


Fig.6-1 General Block Diagram for Discontinuous System

of them at the same time. The usual possible ways to produce discontinuous damping may be listed as

a. Feedback compensation Block 4.

b. Cascade compensation Block 2.

c. Mechanical compensation Block 3.

d. Feedback cascade compensation Block 2 and 4.

e. Feedback and mechanical compensation Block 3 and 4.

f. Cascade and mechanical compensation Block 2 and 3.

g. Compensation by input signal analyser and control computer Block 1 and 5.

The compensation may be produced by mechanical devices or electrical devices or together.

SECTION I - DISCONTINUOUS DAMPING USING FEEDBACK METHOD

Feedback compensation theory is well developed in the design of feedback control systems. Here the main point is to use low order feedback signals to compensate high order systems, and the high order derivative feedback signals are not available.

From the mathematical view, it is nearly impossible to use only low order feedback signals to compensate high order systems, especially in discontinuous systems an overdamped character is needed to brake the system from high speed. In the conventional definition the roots of an over damped system are all real. According to the relation between the value of roots and the value of the coefficients of the differential equation, there will be only one differential equation corresponding to each set of roots. In other words, in a high order system if only the low order feedback signals are available, the only thing that can be done is to change the coefficients of the low order term in the characteristic equation. Common sense indicates that the possibility

of making an overdamped system with real roots is not high.

In the analysis and design of a continuous system, the usual case is to specify a pair of complex roots, they give the required performance of the system such as the damping factor, the overshoot and the settling time. These are called a pair of dominant poles, the assumption is that the other poles are far enough from this pair of dominant poles. This "dominant" terminology may also be applied to the overdamped system, to call one or a pair of real roots dominant poles of the overdamped system provided that the other real roots are far away and the absolute value of any complex roots are large enough. This statement can be proved briefly by using the frequency response approach: For a system with one or two real roots near the origin of the plane, then in the Bode diagram there will be an attenuation of -6 or -12 db slope in the low frequency range, the effect of the other roots will be small since they are far away from these dominant poles even if some are complex. The output of the system will have no overshoot and no high frequency oscillations. This situation is apparent in the design of the fourth order physical system in Chapter II. In this chapter a systematical analysis is made along this line.

In the second order system, the tachometer feedback is enough to make the system overdamped. The larger the tachometer feedback the larger the separation between two real roots, until the main channel gain or the feedback channel gain is saturated.

In third order systems, the tachometer feedback itself may or may not make the system overdamped. In this discontinuous system the accelerated trajectory is usually underdamped. A pair of complex roots of the underdamped characteristic equation may be chosen near the $j\omega$ axis, and a real root far away from the origin of the S plane. In this case the system can be made overdamped by only feeding back the negative tachometer signal. But

if the imaginary parts of the complex poles are large, or the real root is not far away from the origin, then three real roots cannot be obtained by only using tachometer feedback. A typical root locus plot is given in Fig. 6-2. The result indicates that if the absolute value of the real root is equal or larger than five times the imaginary part of the complex poles then the overdamped characteristic equation with three real roots can be obtained; otherwise there will be one small real root and two large complex roots. As mentioned before, the small real root may be regarded as the dominant root and the effect of the large complex roots neglected, but in this case the gain of the feedback signal should be very high. From Fig. 6-2, complex roots tend to approach an asymptote as the tachometer gain goes high and the small real root approaches the origin. The high frequency character will be damped out by the small root. If the real root in the underdamped case is near the origin, then the damping factor of the complex roots will be very small. By using the frequency response concept the real root makes the system have -6 db per octave attenuation, but the complex roots make the system have a positive peak at high frequency. The resultant Bode diagram will be like that sketched in Fig. 6-3. For small values of f, there may be high frequency ripples superposed upon the over damped trajectory. How serious this is will depend upon the application and the character of the system under consideration. In general tachometer feedback can make an underdamped third order system overdamped provided that in the underdamped case the ratio between the absolute value of real roots and the imaginary part of the complex roots is large. The limitation of the overdamped case is caused by the high frequency ripples due to the complex roots.

If the second derivative of the output signal can be obtained, then there is no problem in producing an over damped system with three real roots in the characteristic equation.

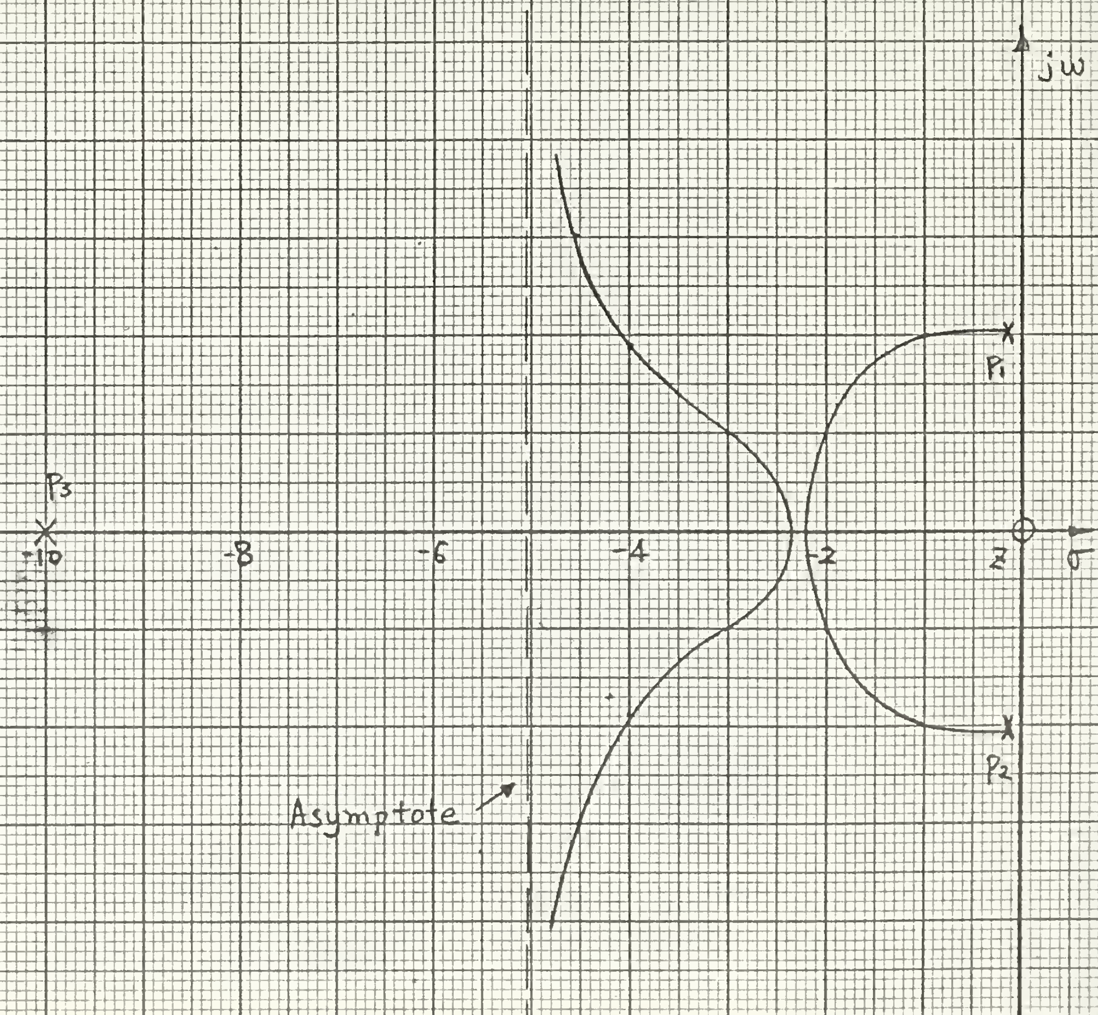


Fig. 6-2 Generalized Root Locus Plot of a Third Order System with Tachometer Feedback Only.

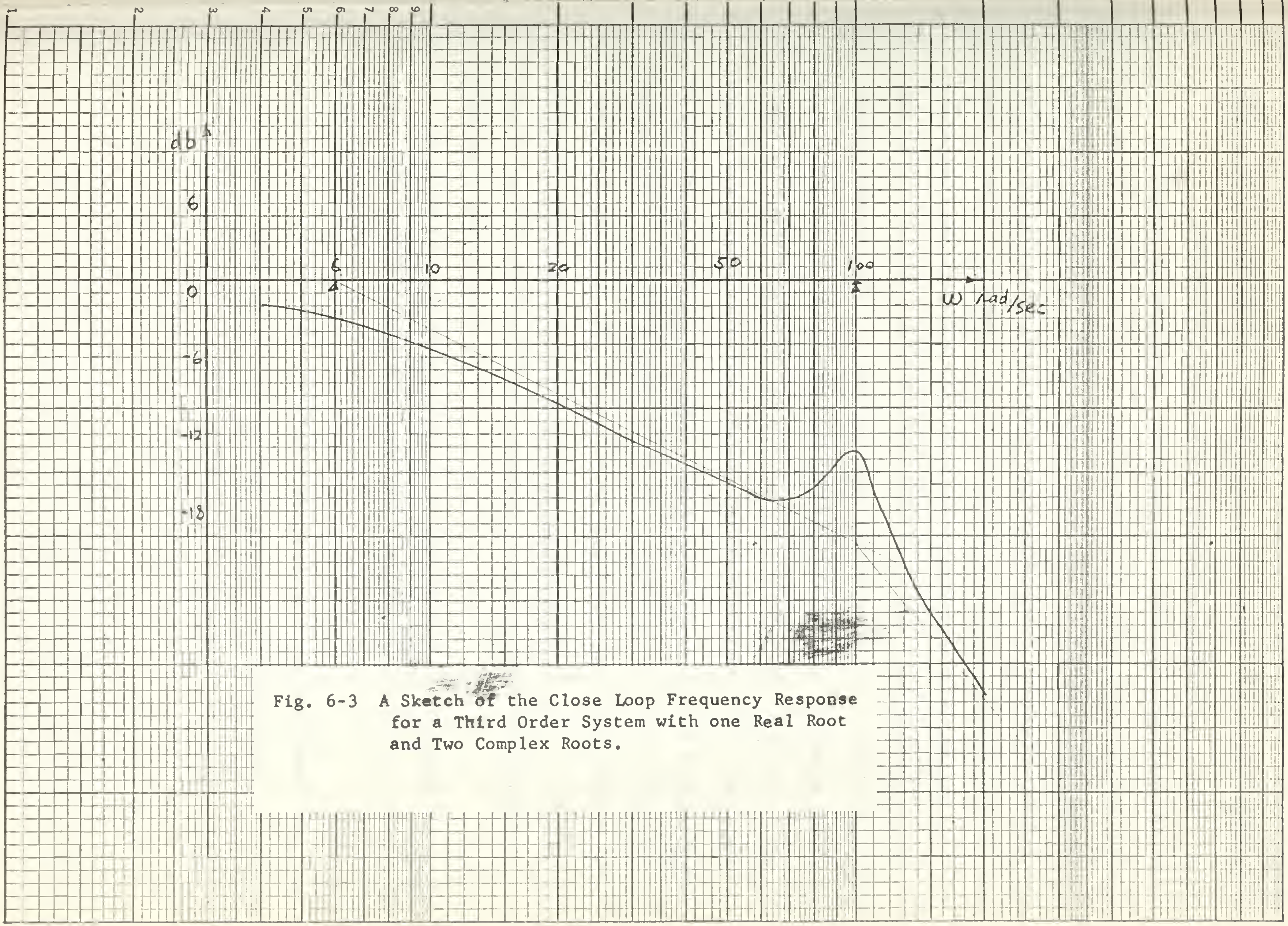


Fig. 6-3 A Sketch of the Close Loop Frequency Response for a Third Order System with one Real Root and Two Complex Roots.

SECTION II - COMPUTER STUDY AND SPACE ANALYSIS ABOUT THIRD ORDER DISCONTINUOUS SYSTEMS WITH COMPLEX OVERDAMPED TRAJECTORY.

In Chapter II the die-out trajectory with hyperplane switching was explained by saying that the initial condition at the switching instant just cancelled the residue of the smallest root, and the die-out trajectory is entirely determined by the large roots. But in the first section in this chapter there is one small real root and two large complex roots for the overdamped case, therefore when the die-out trajectory is determined by this pair of complex roots, the trajectory will stay in a plane decided by the two complex roots. If the switching operation is not in this hyperplane, then the trajectory is decided both by the real root and the complex roots. Since the damping factor of the complex roots decides the character of the die-out trajectory, therefore, the die-out trajectory may have an overshoot if the damping factor is not large. This has to be considered in the actual design problem. In this section instead of making a numerical solution to show the die-out part of a discontinuous system with complex overdamped decelerated trajectory the analog computer is used to do this job, with the same set up the switching time is changed to see how the trajectories are going on in error space.

By using the same computer set up as that in Part II of Chapter II, only changing the overdamped case to have complex poles, the following equations obtain:

$$\dddot{E} + 2.02\,\ddot{E} + 0.54\,\dot{E} + E = 0 \qquad \text{(Underdamped)}$$

$$\dddot{E} + 4.2\,\ddot{E} + 5.8\,\dot{E} + E = 0 \qquad \text{(Overdamped)}$$

$$r_1 = -0.2\,, \quad r_2 = -2 + j1\,, \quad r_3 = -2 - j1$$

The equation for the control computer

$$u_1 = (r_2 \times r_3)\dot{E} + (r_2 + r_3)\dot{E} + \ddot{E} \qquad \text{(Take r's as positive values)}$$

$$= 5E + 4\dot{E} + \ddot{E}$$

$$= 5\bar{x} - 4\bar{\dot{\theta}}_c - \bar{\ddot{\theta}}_c$$

Computer setting values :

$$w_4' = 0.7 \quad \text{or} \quad w_4' = 0.263$$

$$w_5' = 1.93 \qquad w_5' = 1.75$$

$$w_{11} = 0.75$$

$$w_{12} = 0.4$$

$$w_{10} = 0.05$$

The computer recording by using different switching time is in Fig. 6-4. By using Fig. 6-5a,b,c, a space model can be made.

By changing the coefficients of the over damped equation to make the small real root to be larger and the complex root to be smaller as in the following computer set up:

$$\dddot{E} + 2.5\ddot{E} + 3\dot{E} + E = 0$$

$$r_1 = -0.5, \quad r_2 = -1 + j1, \quad r_3 = -1 - j1$$

$$u_1 = 2E + 2\dot{E} + \ddot{E}$$

Computer setting values :

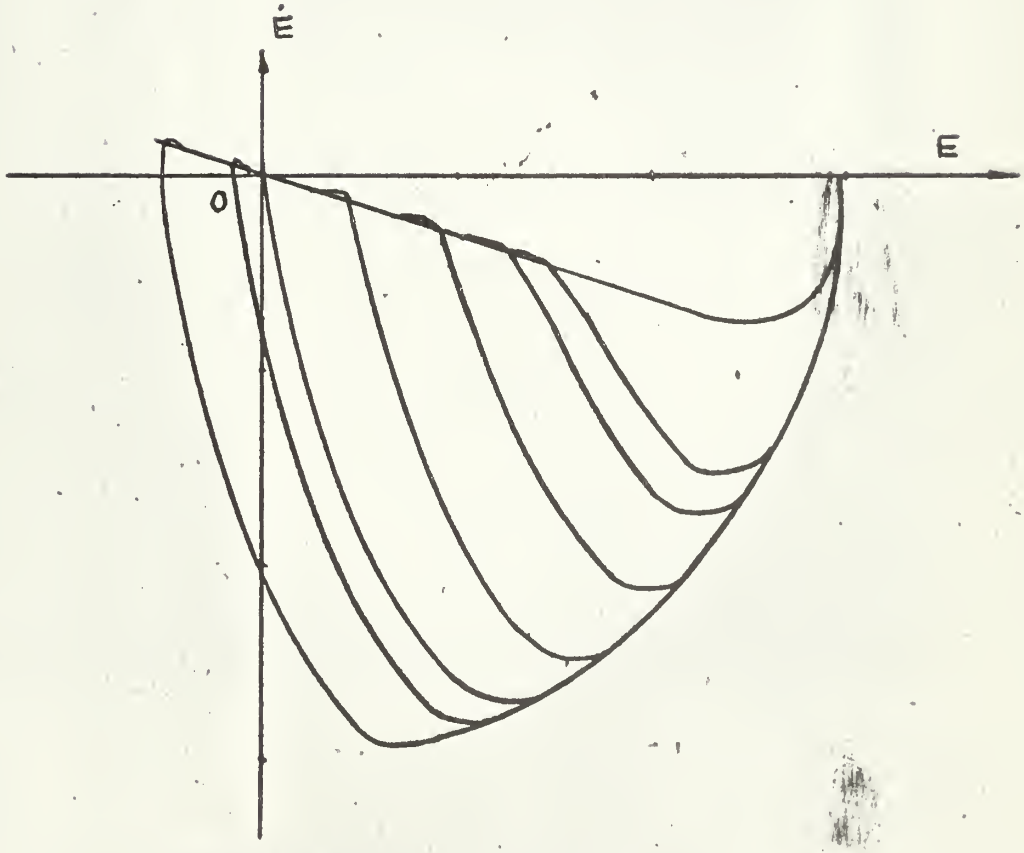
$$w_4' = 0.416 \quad \text{or} \quad w_4' = 0.079$$

$$w_5' = 1 \qquad w_5' = 0.82$$

$$w_{10} = 0.1, \quad w_{11} = 0.6, \quad w_{12} = 0.4$$

The recordings for various magnitudes of step input are in Fig. 6-6a, b, c, and a space model can be constructed as before.

Fig. 6-4 Computer Plot for a Second Order Discontinuous System with Complex Over damped Die out Trajectory and Switched Earlier or Later



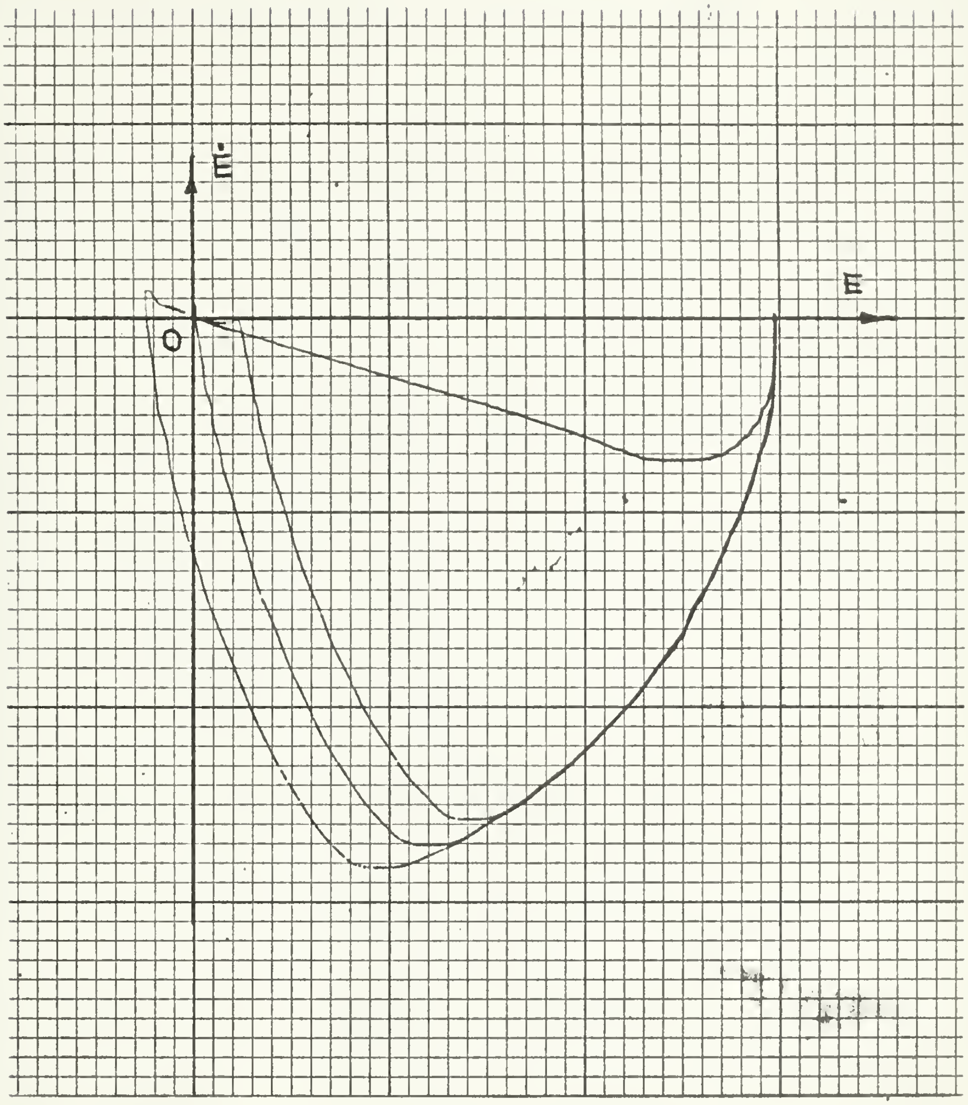


Fig. 6-5 Computer Plots for Making an Error Space Model
(a) E vs Ė Plot

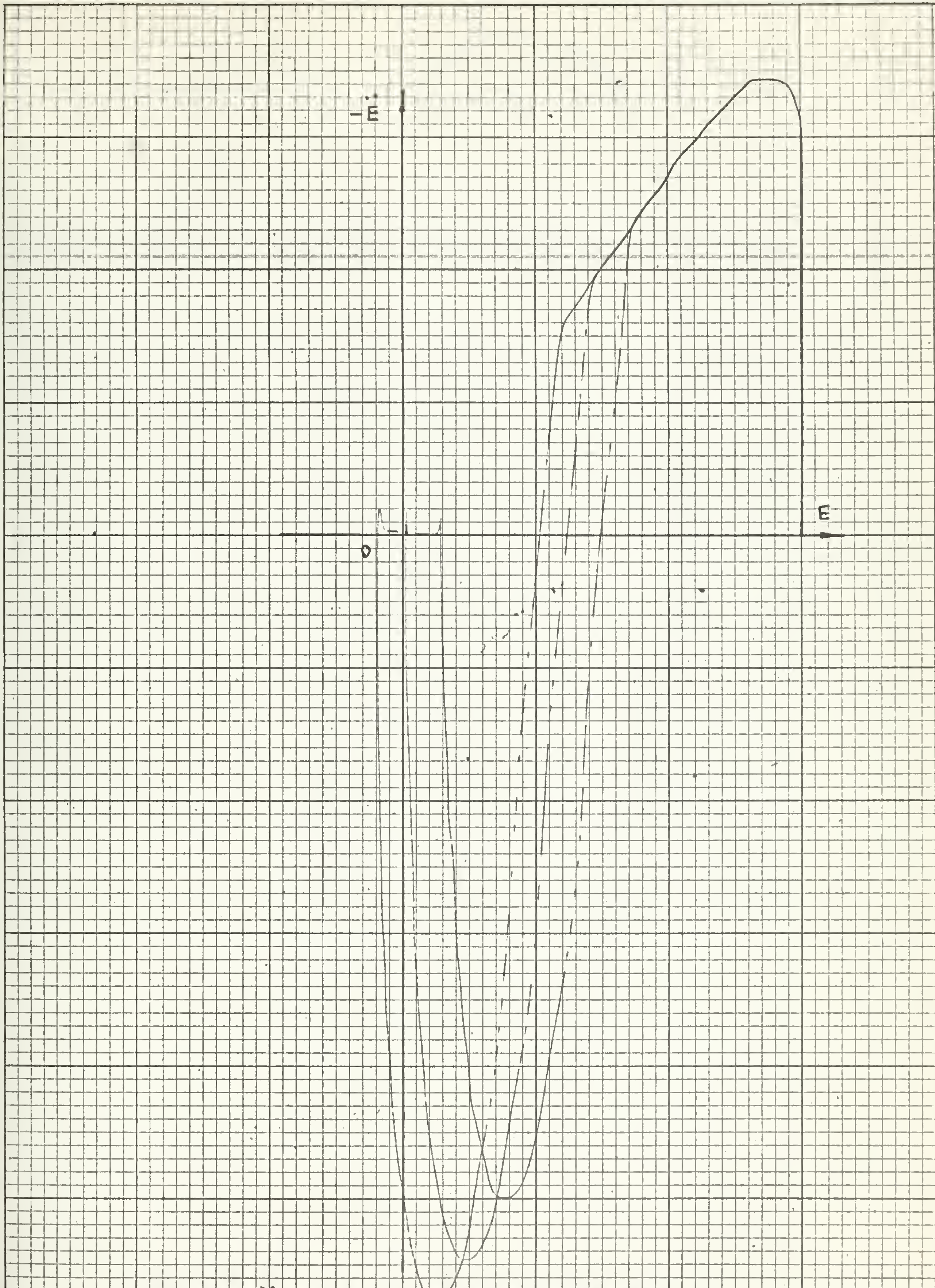


Fig. 6-5b $\ddot{E}$ vs E Plot for Error Space Model

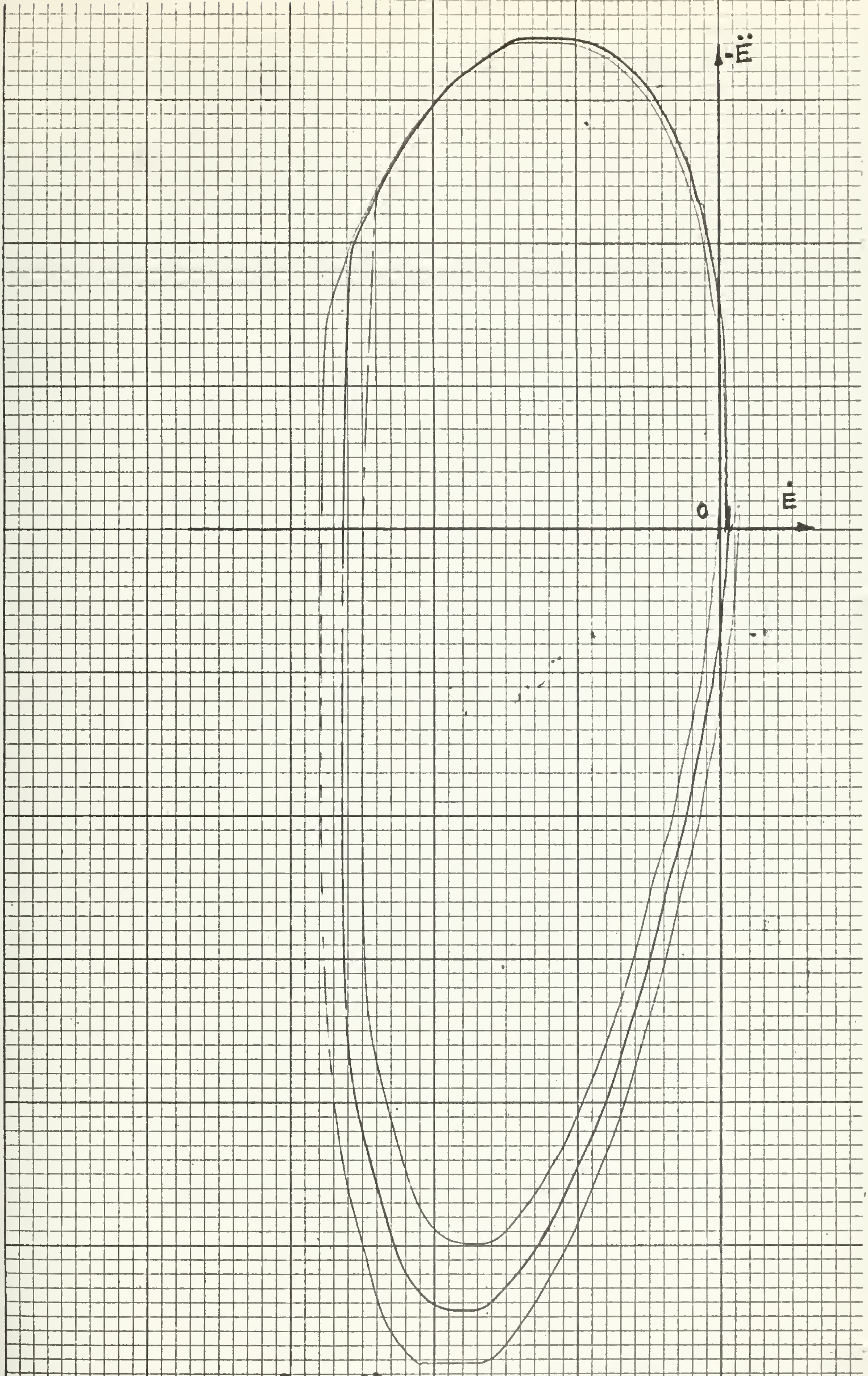


Fig. 6-3c $\dot{E}$ vs $\ddot{E}$ Plot for Error Space Model

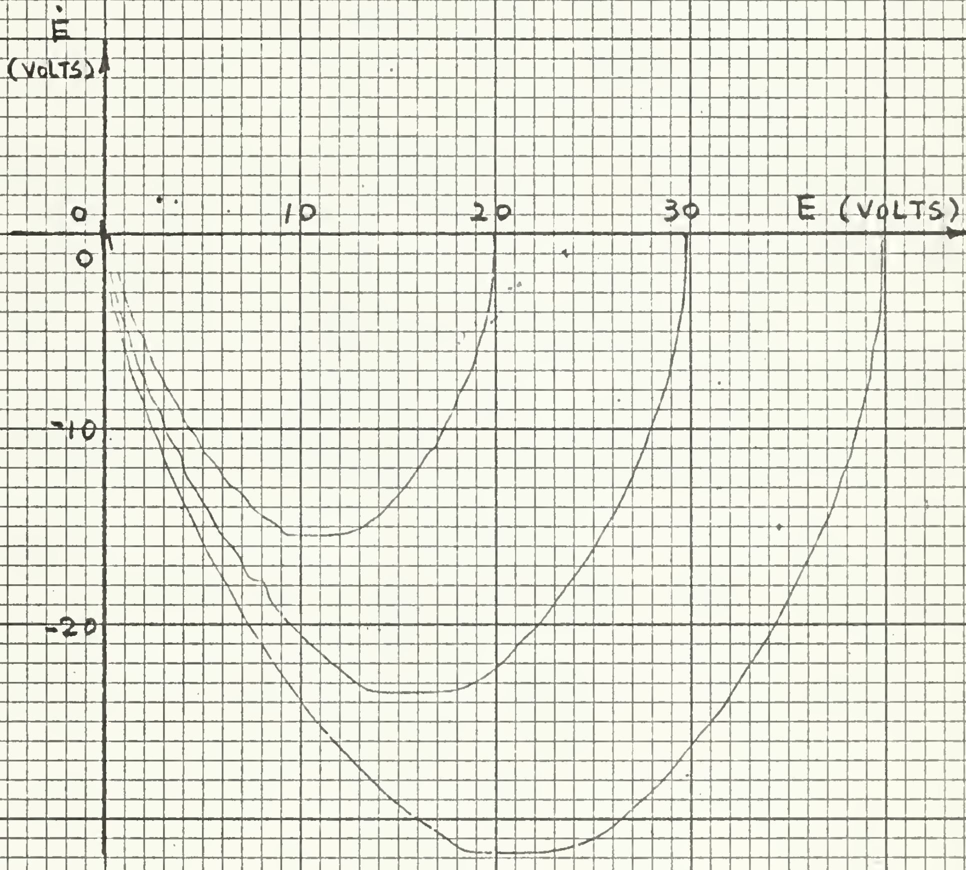


Fig. 6-6 Computer Plots for Various Magnitude of Step Inputs
(a) $\dot{E}$ vs E Plot

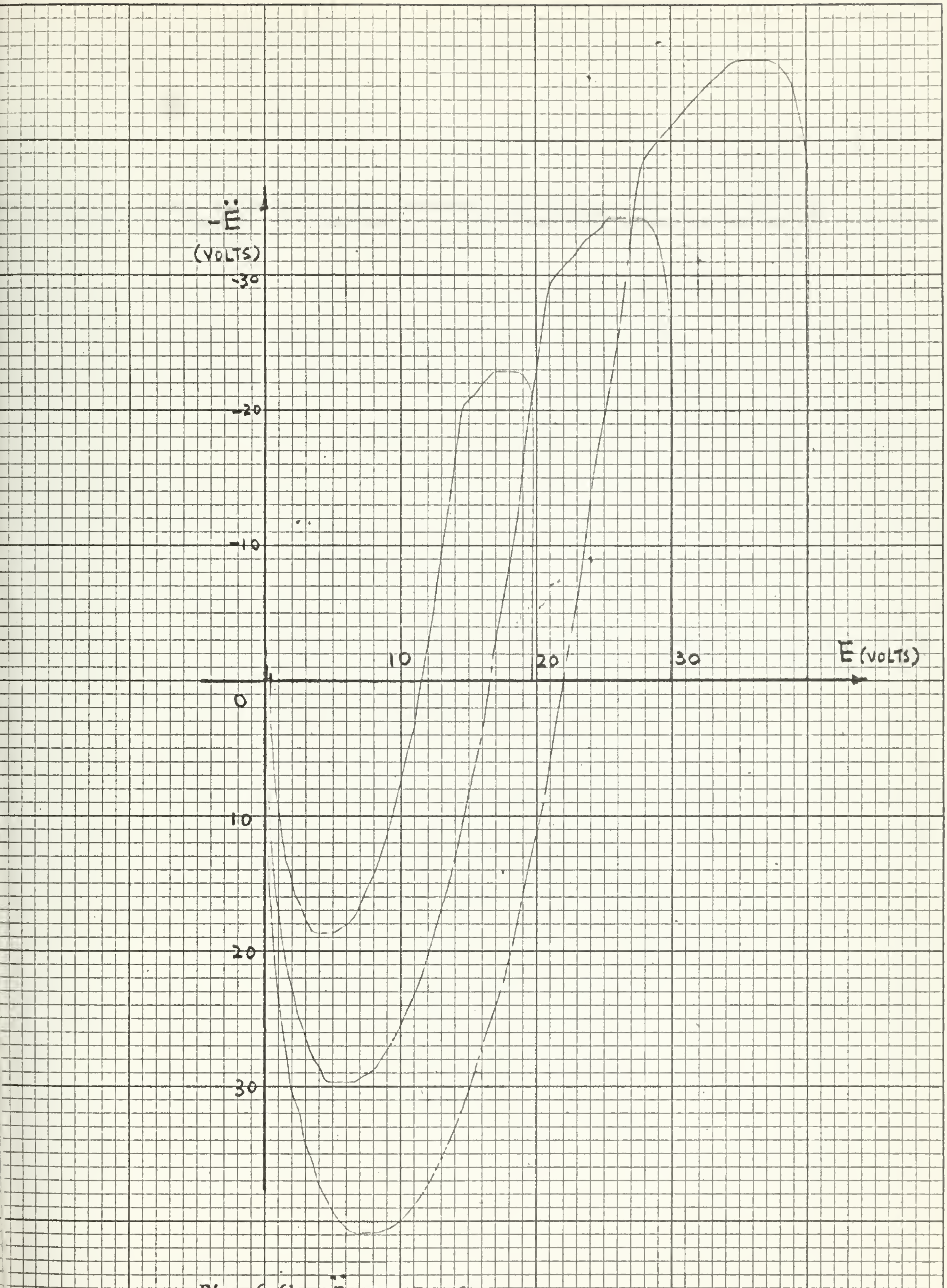


Fig. 6-6b $\ddot{E}$ vs E Plot

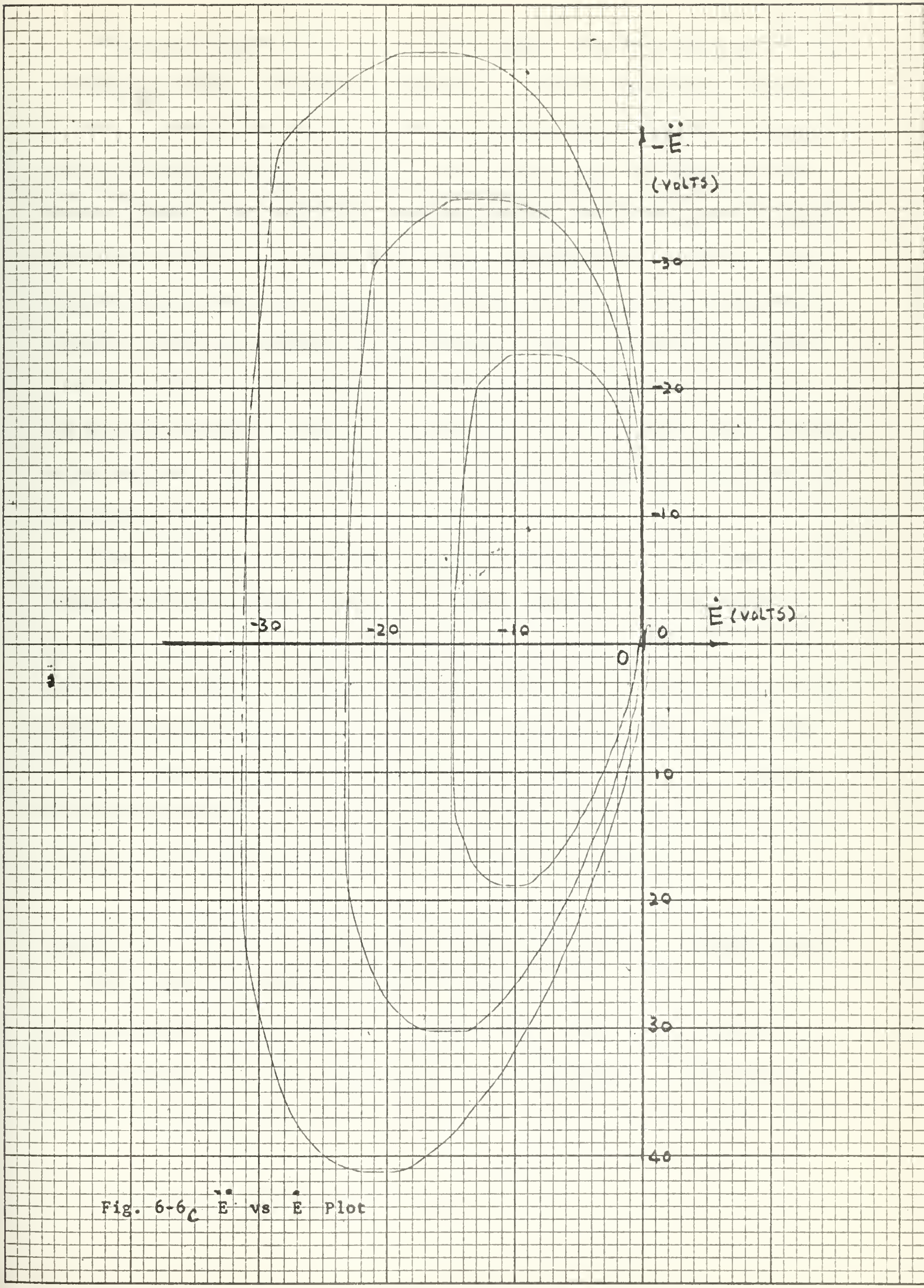


Fig. 6-6$_C$ $\ddot{E}$ vs $\dot{E}$ Plot

From the recorded plots in the above figures the characters of the die-out trajectory may be described in general, a complex overdamped hyperplane is used to control the system, by using the phase space concept. The general character is nearly the same as that using a real overdamped discontinuous system. The trajectory, when switched at the proper point, will go through the origin for various magnitudes of step inputs. When the switching operation is earlier or later, the trajectory will go to the overdamped trajectory which corresponds to the continuous case, and oscillate around this slow trajectory as in Fig. 6-4. The residue of the small real root has not been cancelled out by the initial condition, the small real root will be the dominant root in this case, and the small oscillations are caused by the complex roots.

From Fig. 6-5b and Fig. 6-6c a particular character is indicated by the trajectory near the origin that is the small velocity overshoot caused by the complex poles. From the space analysis discussions in Chapter I the underdamped trajectory tends to confine its later part to a plane and oscillate in the quadrants $4', 4, 3, 2, 2', 1', 4'$. This plane has an angle with the $\ddot{E}$ vs $\dot{E}$ plane, therefore an overshoot that looks large in $\ddot{E}$ vs $\dot{E}$ plane will become smaller in the E vs $\dot{E}$ plane and even looks like a pure velocity overshoot. This character is preferable in some applications.

SECTION III - HIGH ORDER SYSTEMS COMPENSATED WITH LOW ORDER FEEDBACK SIGNALS.

The commonly accepted concept of tachometer feedback is just like a friction applied to the system. One might think that for any system if a large friction is applied, the system will be overdamped. Unfortunately this is not true for high order systems. For example, in the fourth order

physical system in Part II of Chapter II, if only tachometer feedback is used the system will be unstable if the gain of feedback is too high (Fig. 6-7). Therefore, a fourth order system cannot be made overdamped by using tachometer feedback only.

By using tachometer output and its first derivative feedback to compensate a fourth order system, the discussion in Chapter II has given some ideas about how to adjust the gains of the feedbacks for that particular system. Here an attempt is made to analyze the methods for some other cases, for the purpose of obtaining a general view of the compensation technique.

Since two complex roots near the imaginary axis are decided by the design of the underdamped system, assume the other two roots of the fourth order characteristic equation are on the negative real axis as in Fig. 6-8a. The possible ways for obtaining an overdamped system are illustrated in Fig. 6-8b, c.

Since the general equation for these root locus plots is

$$\frac{KK_a(S + K_T/K_a)S}{(S+r_3)(S+r_4)[S^2+(r_1+r_2)S+r_1r_2]} = -1$$

Therefore, there are many possible cases due to the different position of the movable zero which is decided by the ratio K_T/K_a. In Fig. 6-8b by using the general character of the root locus the root locus tends to go to the real axis and then separate into two parts to terminate at the zeros. Two real roots can be found on the real axis if the gain is high enough. Also, move the movable zero to the left as far as possible, because this gives a chance to get only one small root for hypersurface switching. But the movable zero cannot go over the smallest real root of the underdamped characteristic equation, otherwise large complex roots appear as in Fig. 6-8c. The easiest way to design this kind of system is to put the movable zero at the position where the smallest real root of the underdamped characteristic

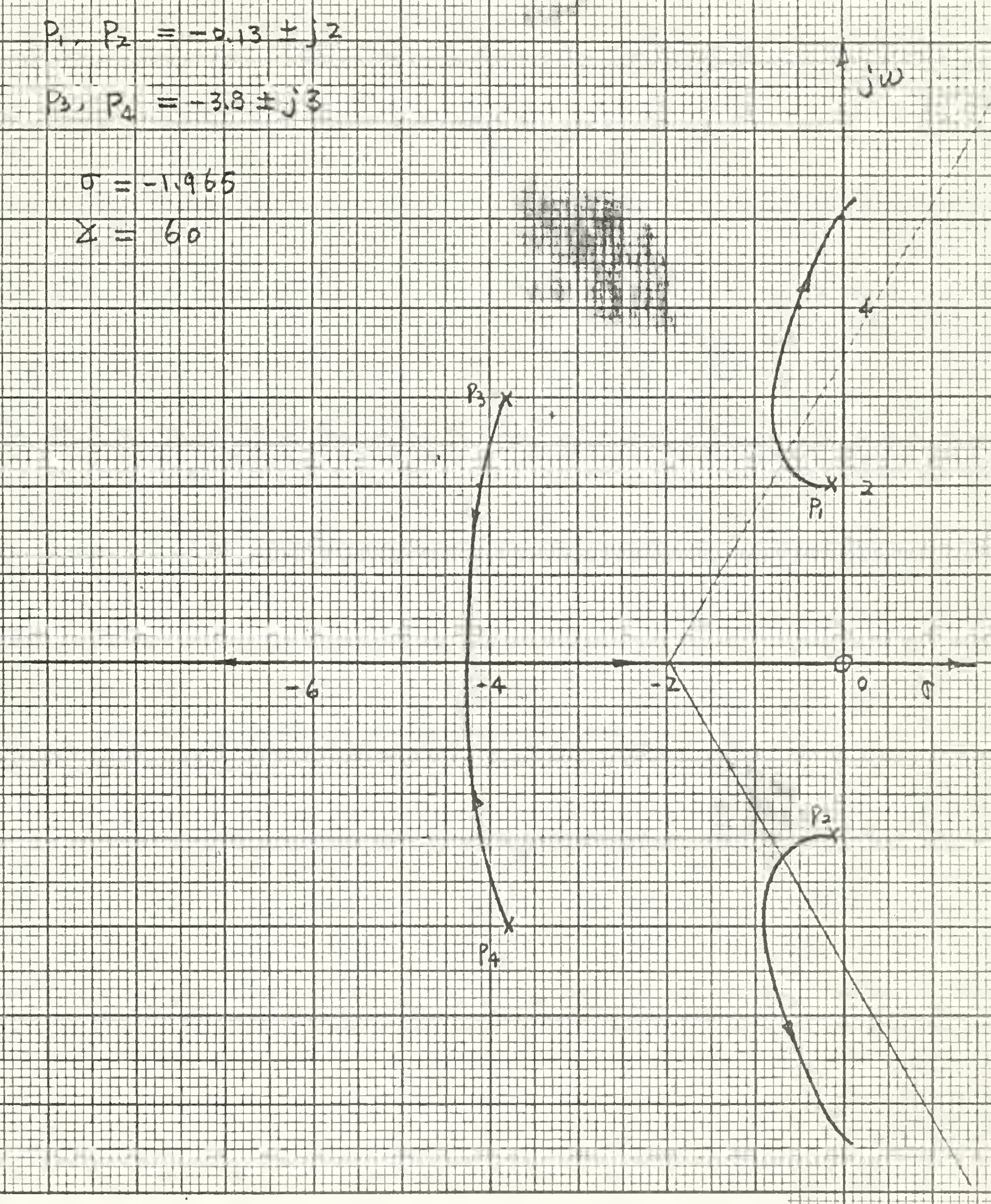


Fig. 6-7 Root Locus Plot for a Fourth Order System with Tachometer Feedback Only.

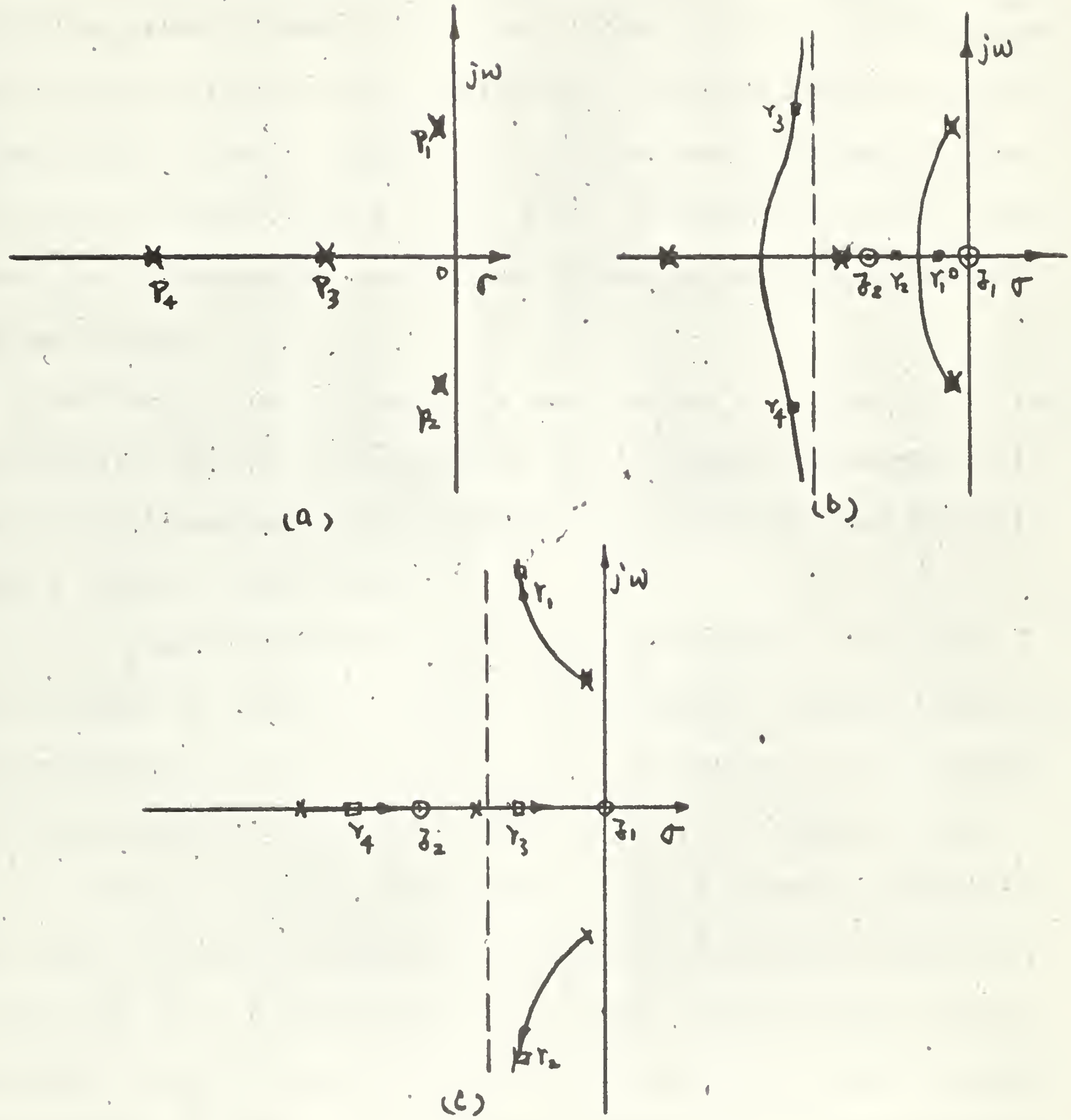


Fig. 6-8 Root Locus Sketches for a Fourth Order System with Velocity and Acceleration Feedbacks.
(a) Poles of Underdamped close loop transfer Function.
(b), (c) Possible cases of compensation.

equation is located, then the root locus plot becomes a third order case. But remember that a real root of the overdamped system is already located at the superposed position. A pole cancelled by a zero does not mean the order of the compensated system is reduced by one, unless this zero is in the open loop forward channel, otherwise a common factor in the overdamped characteristic equation can be factored out. Therefore a real root of the overdamped characteristic equation is located at the superposed position. A third order example is given in Fig. 6-9a. The variations of the locations of the roots when the movable zero is near either side of the pole are given in Fig. 6-9b, c.

In the fourth order system, since two zeros are to be used in the root locus analysis, then the system will be stable, because the asymptote is always vertical, provided that the movable zero is not far away from the origin or from the largest pole.

If the original under damped system has four complex roots as the physical system in Chapter II, the best way to get an overdamped system is to put the movable zero at the position which corresponds to the attenuation factor of the pair of complex poles* not near the imaginary axis. As long as they are not near the imaginary axis, an applicable overdamped system can be obtained by using only acceleration and velocity feedbacks.

When the order of the system is more than 4, from the angle between the asymptote and real axis the system will be unstable if only these two feedback signals are used. But the general approach will be the same as that used in fourth order systems, especially when the poles are not near the imaginary axis, a fourth order approximation in some cases would be considered as acceptable.

*The roots of the underdamped characteristic equation are the poles in the equation of root locus plot for overdamped feedback control system.

$$(S+7)(S^2+0.4S+9.02)+KK_a(S+7)S=0$$

$$\frac{KK_a(S+7)S}{(S+7)(S^2+0.4S+9.02)}=-1$$

$K=100$

$KK_a=7.68 \qquad k_a=0.0768$

$\frac{k_T}{K_a}=7 \qquad k_T=0.538$

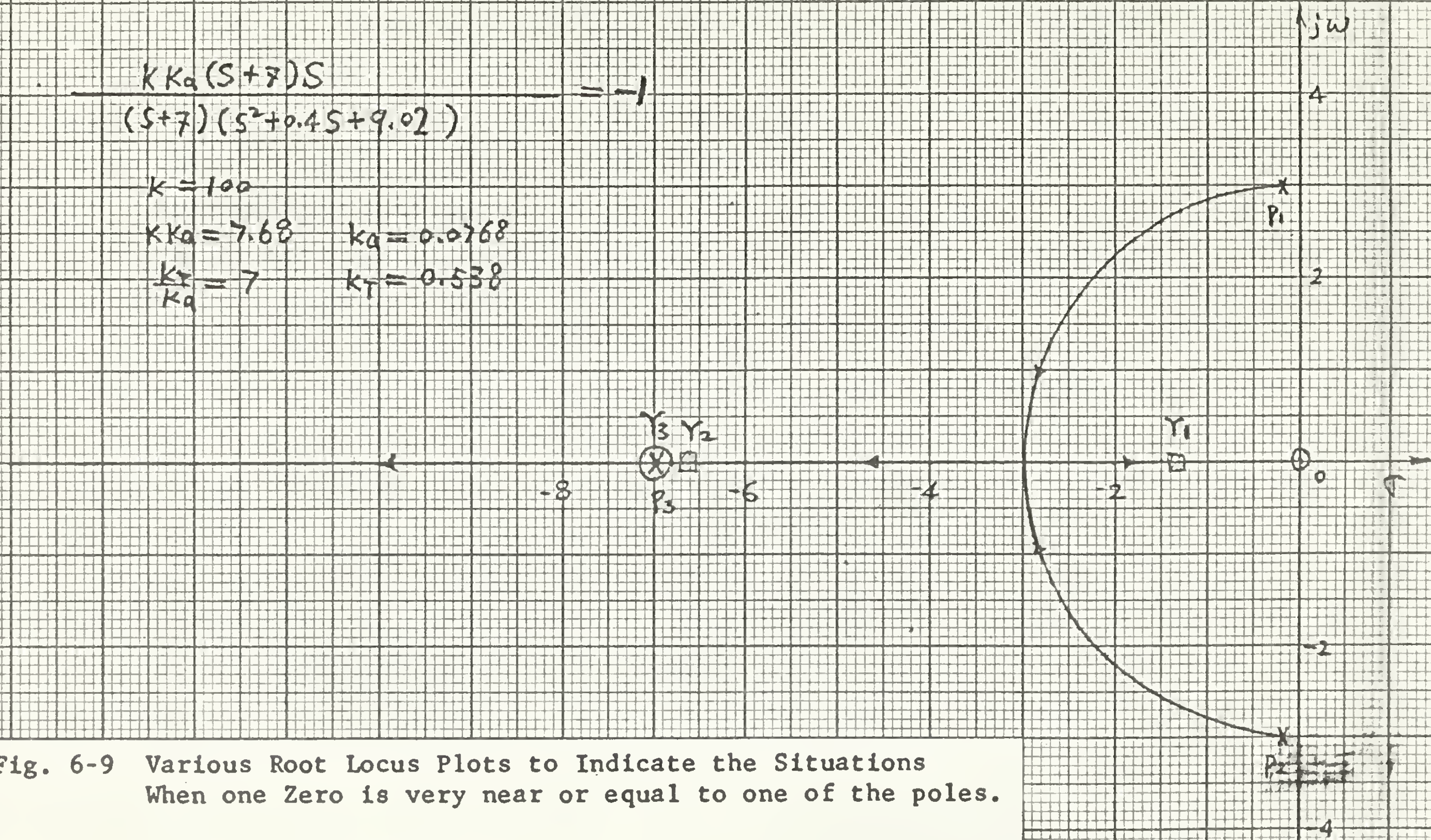
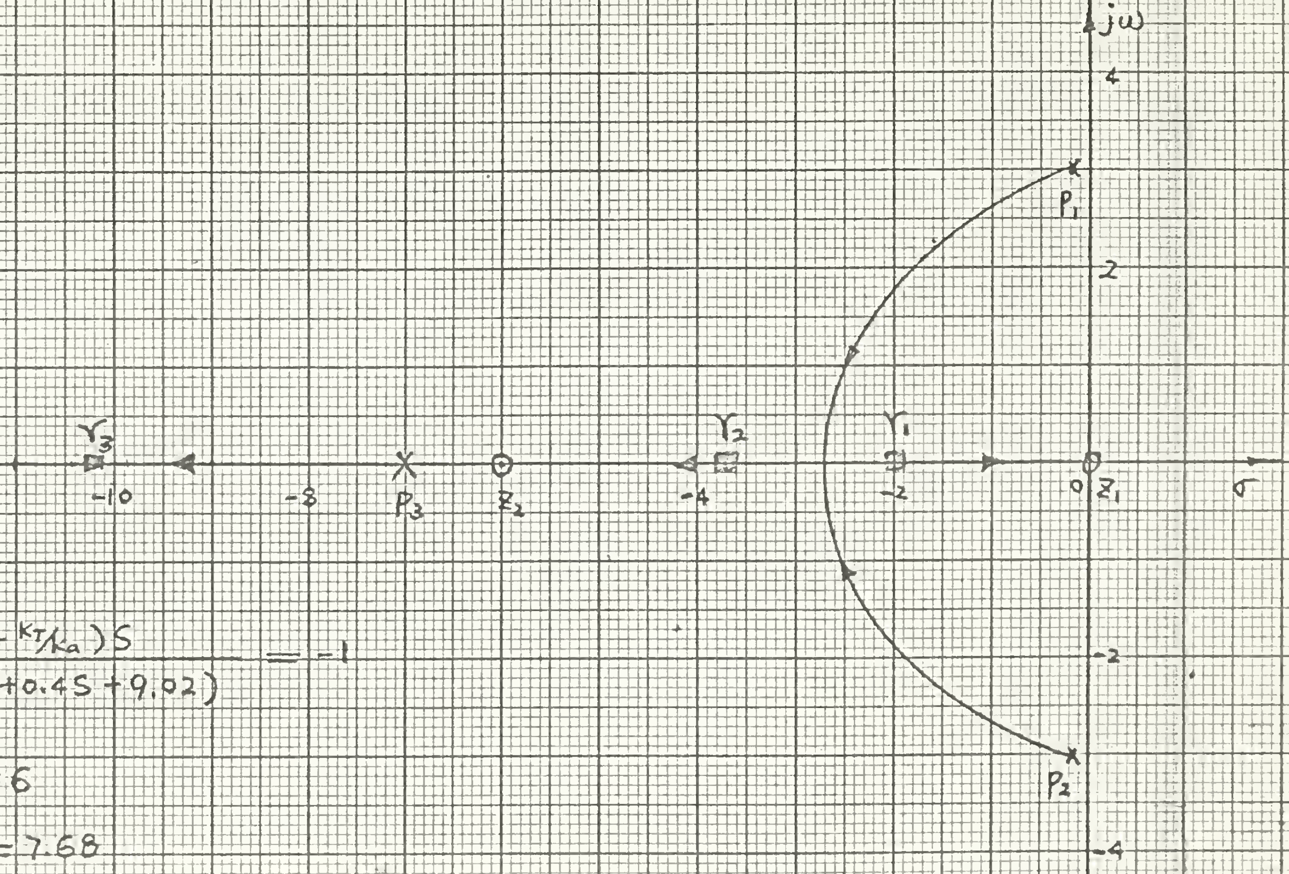


Fig. 6-9 Various Root Locus Plots to Indicate the Situations When one Zero is very near or equal to one of the poles.

(a) Zero equal to pole.



$$\frac{KK_a(S + K_T/K_a)S}{(S+7)(S^2+0.4S+9.02)} = -1$$

$$\frac{K_T}{K_a} = 6$$

$$KK_a = 7.68$$

$$K = 100 \ , \ K_a = 0.0768 \ , \ K_T = 0.46$$

Fig. 6-9b One Zero is near a pole. ($|P_3| > |Z_2|$)

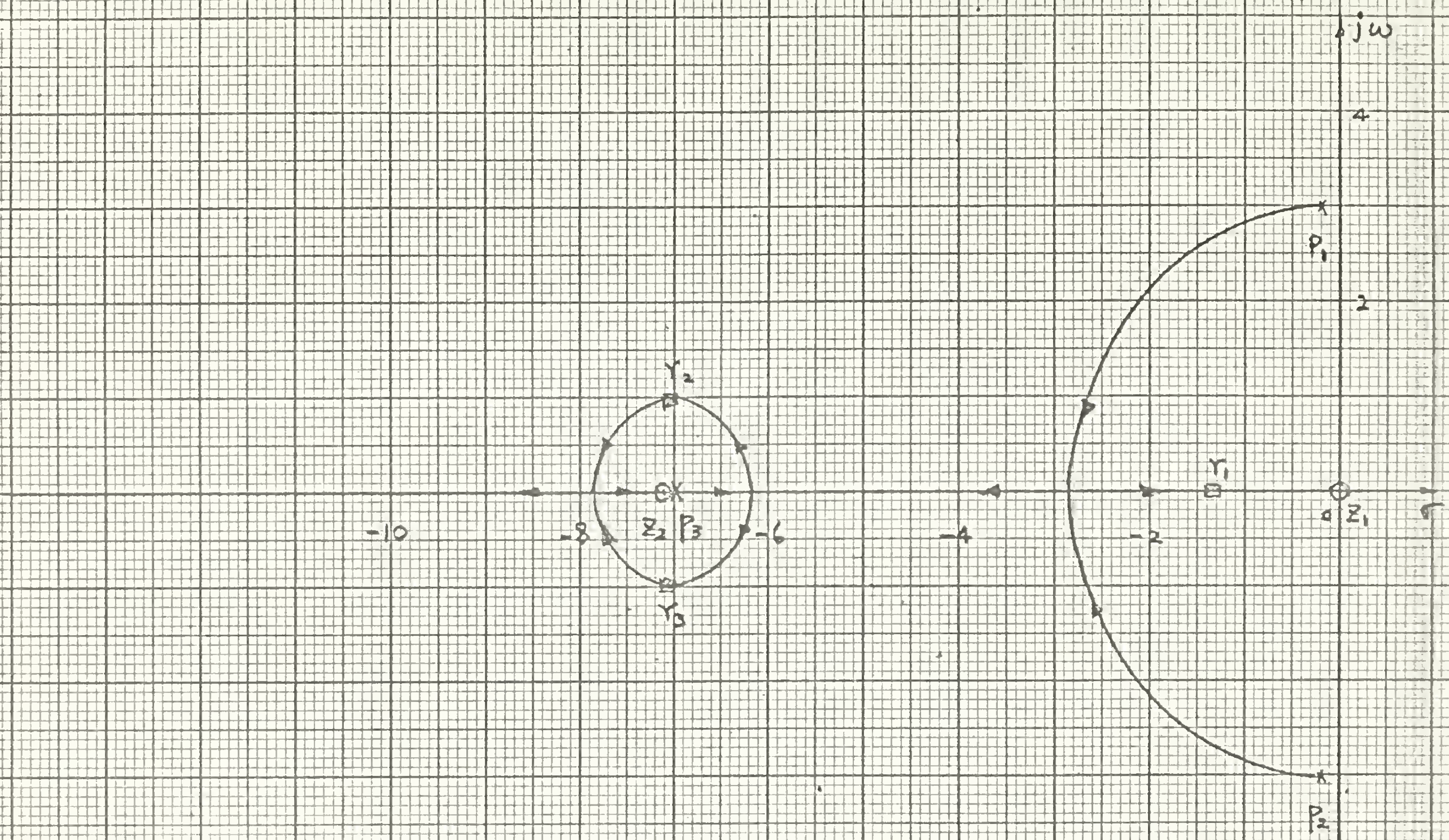


Fig. 6-9c One Zero is near a pole. $(|P_3| < |Z_2|)$

SECTION IV - CASCADE AND FEEDBACK COMPENSATION

The discussion in this section begins with Block 2 in the general diagram. The generally used theory in cascade compensation considers the lag or lead compensator or their combinations. Here the considerations when used in a discontinuous system have to be investigated. It is well known that by using lag, lead or lag and lead networks roots of the characteristic equation can be placed at desirable positions in the S plane. In other words, the system can be made underdamped or overdamped by this kind of cascade compensation. According to the theory of discontinuous systems a proper switching computer may be found to switch the networks in or out, or to change from lead to lag or vice versa. An investigation has been made by Skerrett in his Masters Thesis (1959). The general approach can be illustrated as in Fig. 6-10.

Block N_1 and N_2 are different kinds of networks. The initial conditions are to be made as nearly the same at contact point 1 and 2, the change of gain due to the different attenuations in N_1 and N_2 is assumed compensated by themselves or the gain of the amplifier can be controlled. The computer results illustrated that for such compensation fast and dead beat response cannot be obtained as in feedback compensated systems, because in the above assumptions the initial conditions on part 1 and 2 are nearly the same at the switching instant, then this is a continuous change rather than a reversed signal to apply to the amplifier as in the discontinuous feedback system. On the other hand even if the initial conditions are not the same at point 1 and 2 at the switching instant, the character of the network makes the initial conditions limited both in polarity and magnitude. The switching operation can only change the input to the amplifier from a large value to a small value, the overdamped system is controlled by the slow response character

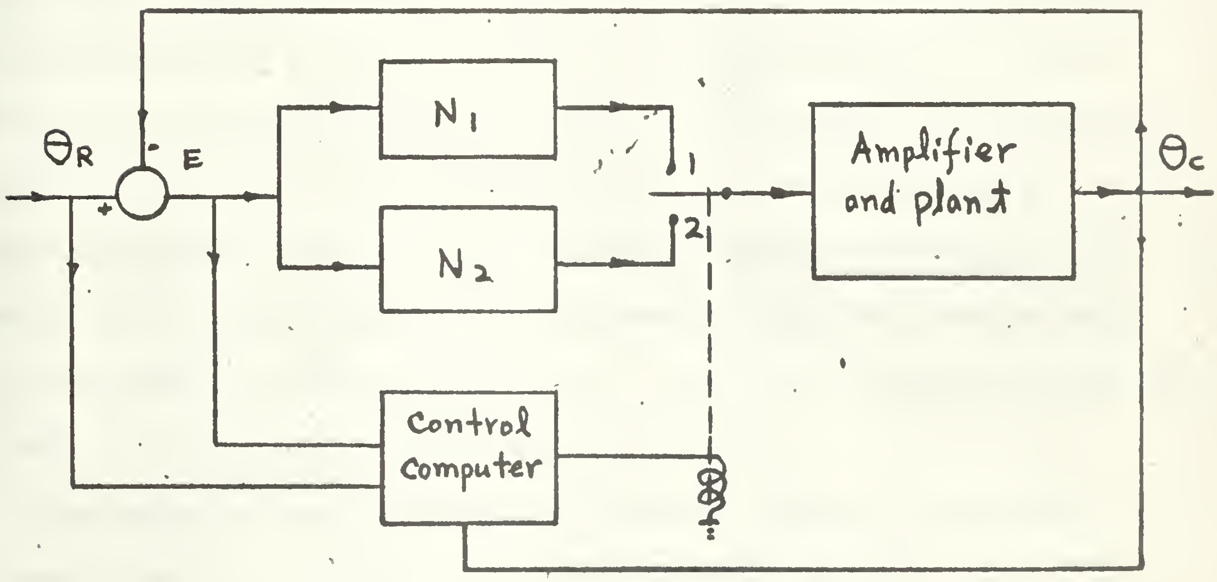


Fig. 6-10 General Block Diagram of Discontinuous System with Cascade Compensators.

(the character of smallest root). Therefore, this is not a switching operation according to the hypersurface which uses initial conditions to cancel the residue of the smallest real root. How to improve the response character by this discontinuous cascade damping method is still a problem that hasn't been solved, one possible approach is the design of networks N_1 and N_2 as active networks.

Cascade compensation together with discontinuous feedback compensation may become very useful especially for high order systems. As indicated before, with a system up to five or higher order, a good overdamped character cannot be obtained for applying the discontinuous theory using only low order feedback signals. But with cascade compensators it may be possible to relocate the undesirable pole or zeros to make a system with dominant real roots. Again there is the same problem of initial conditions, but here the feedback signals provide a part of the driving signals, the deviation from the theoretical trajectory in the die-out part may be in the tolerable range, and at least this is one way to improve the character of high order systems using low order control or compensating signals. Of course the design of a switching computer is based on the theory of approximation.

Since there are no calculations or computer results to confrim the last statement, therefore it may be regarded as a suggestion to the reader for making further investigation.

SECTION V - DISCUSSIONS ABOUT THE DAMPING PROBLEMS IN THE PLANT

The basic idea in this section is to change the character of the system by changing the parameters in the plant (Block 3 in the general block diagram). Several methods have been used in control systems, such as the viscous damper, mechanical braking, shorting the armature circuit, dynamic braking, and stored energy braking. The general purpose is to provide a better die-out trajectory, to make it a straight line near the $\dot{E}$ axis in phase plane or near the $\ddot{E}$ vs $\dot{E}$ plane in phase space. From the mathematical view the damping effect produced by the methods mentioned above is equivalent to increasing the tachometer feedback in feedback control systems. But this damping effect is applied only to the plant, therefore, there is no saturation problems as that encountered in the feedback control systems. Especially for relay servo systems, since the feedback signals can only control the switching trajectory or hypersurface, they cannot change the character of the trajectory, then the damping device applied to the plant will improve the response. In some papers the investigation about how to control a plant with complex poles had been made. The results indicate that the control computer needs to be very complex. (Lotz 1960, Chandaket and Leondes 1960). Therefore, the approach of using a damping method in the plant will give better results provided that the damping can be applied to the plant.

On the other hand, since the damping effect is needed only in the die-out part of the trajectory, then the discontinuous method will give the best results. The theory of how to apply the damping is different due to the method which is used to produce the damping. But the general theory about when to apply the damping is just the same as discussed in the previous chapters. For high order systems the damping or braking method should also be able to produce damping effects according to the higher order derivatives of the system

output. Up to now this is not easy to do. Using cascade or feedback compensation combined with the damping effect in the plant may be a better way to solve this problem.

In the general block diagram of discontinuous systems, it is possible to use the control computer to control the time of throwing in the damping effect and the character of the damping. Therefore, if a device is designed making the damping effect vary according to the output of the control computer it may be possible to find an instrument which gives a high order damping effect. The basic diagram is suggested in Fig. 6-11. It may be to design a braking device with braking torque proportional to the error and its derivatives, such as a braking motor with field controlled by an amplifier, the amplifier being a summer of the control signals. For example the simplest case is an electric damper, its armature circuit is shorted, its field is controlled by a relay to connect or disconnect it from the output of the computer. The switching time is also controlled by the control computer. The simplified diagram is given in Fig. 6-12. The voltage applied to the electric damper may be only taken from a D. C. source.

SECTION VI- ANALYSIS AND DESIGN ABOUT THREE MODE SYSTEMS

By looking at the general block diagram of discontinuous system and combining the ideas of linear servo, relay servo and dynamic braking together, it may be possible to design a three mode system which gives a better response character. The general features of these three modes are:

Mode 1: Underdamped accelerating trajectory.

Mode 2: Heavily damped die-out trajectory.

Mode 3: Overdamped character near the origin.

Three of the possible combinations are given in Fig. 6-13, 6-15, 6-16.

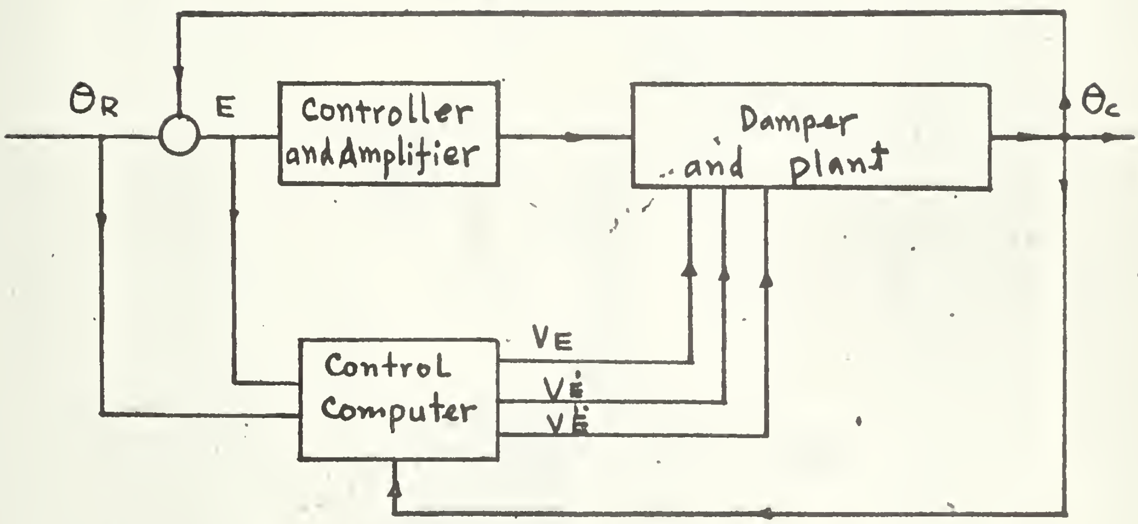


Fig. 6-11 Suggested Block Diagram for Damping Device.

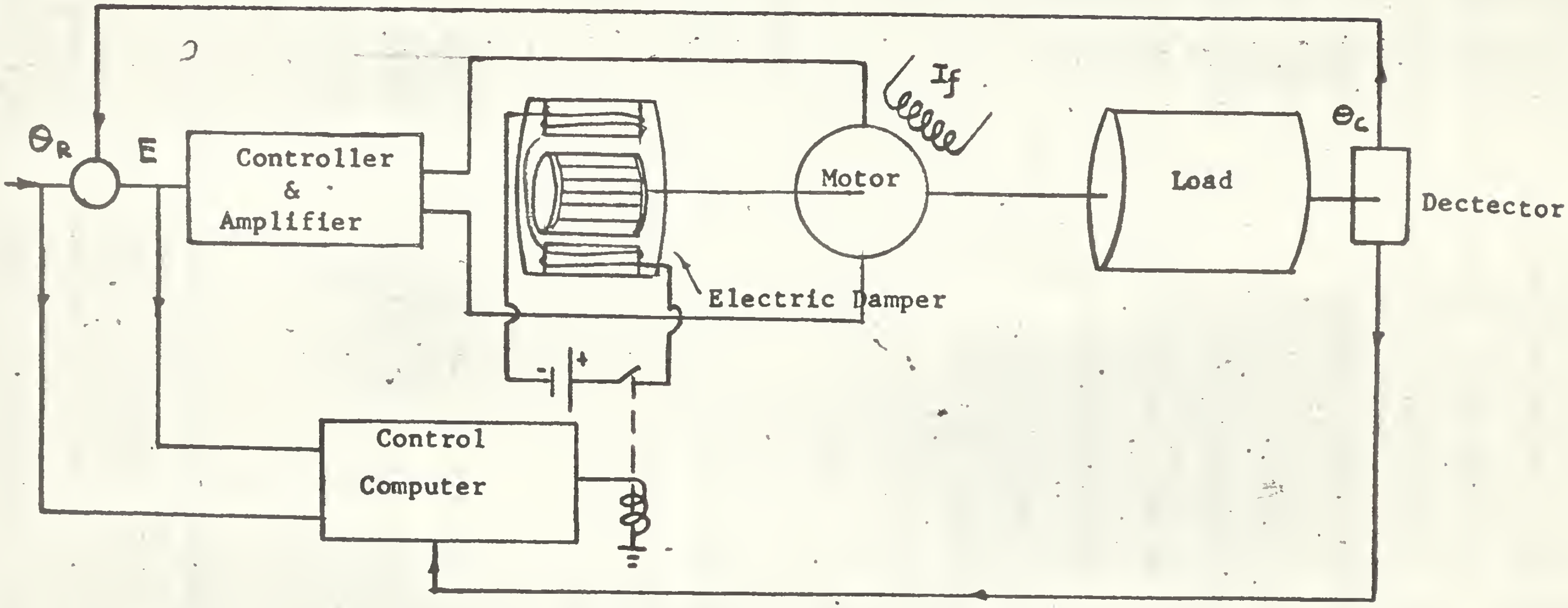


Fig. 6-12 Simplified Diagram for a Discontinuous System with Electric Damper.

A brief discussion for each case will be given separately.

<u>Case 1: Relay servo with dynamic braking and linear auxiliary motor.</u>

In Fig. 6-13 M_1, is an auxiliary motor with high gain but smaller in size. It is controlled by the output of the amplifier. M_2 is the main relay servo motor. Its applied voltage is controlled by the polarity of the amplifier. But in the dead zone of the relay, M_2 is under dynamic braking condition by the shunt resistor R. The switching line or surface is controlled by the feedback block $H(S)$. In the accelerating condition, M_1 and M_2 both are driving in the same direction. The whole system is in an underdamped condition which gives a fast rising character. In the die-out part of the trajectory the switching action is designed to make the whole die-out trajectory stay in the dead zone of the relay, then the system is in a braking condition. The trajectory will be straight and near the $\dot{E}$ axis. At the last part of the trajectory, the auxiliary motor will drive the system to the origin. The motion will be slow, but the final steady state error is zero. A simple relay may be added to disconnect the braking resistor when the error is near zero, to make the last part of the trajectory overdamped but not heavily damped. The typical response curve for a step input is given in Fig. 6-14.

The stored energy braking theory can also be applied to replace the braking resistor , the general performance of the whole system is dominated by a relay servo character, the improvement is to use the auxiliary motor to correct the steady state error.

Case 2: <u>Continuous system with relay servo accelerating and braking.</u>

In Fig. 6-15, M_1 is larger than M_2 the character of M_2 is to accelerate and damp the continuous system. If a resistor is used to control the braking character of M_2, then this is equivalent to discontinuous tachometer feedback. But M_2 can increase the main channel gain of the system

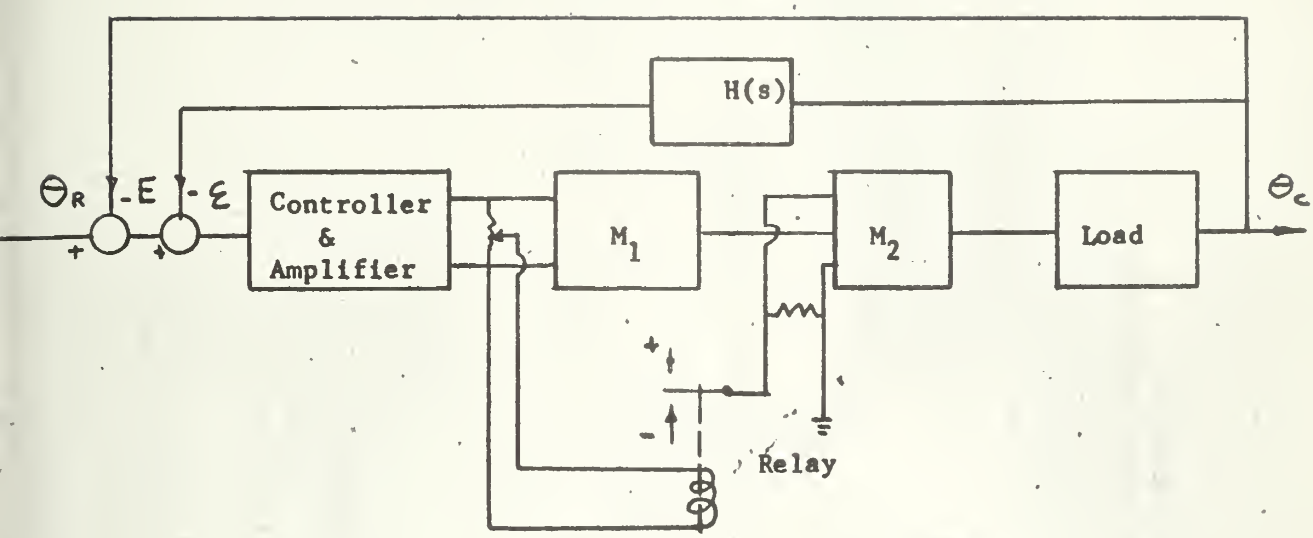


Fig. 6-13 Block Diagram of a Relay Servo with Dynamic Braking and Linear Auxiliary Motor

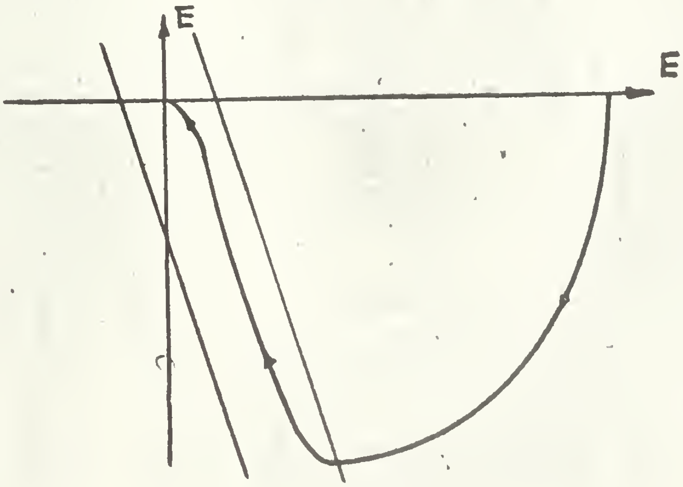


Fig. 6-14 Typical Response Curve 66rThree Mode System for a Step Input.

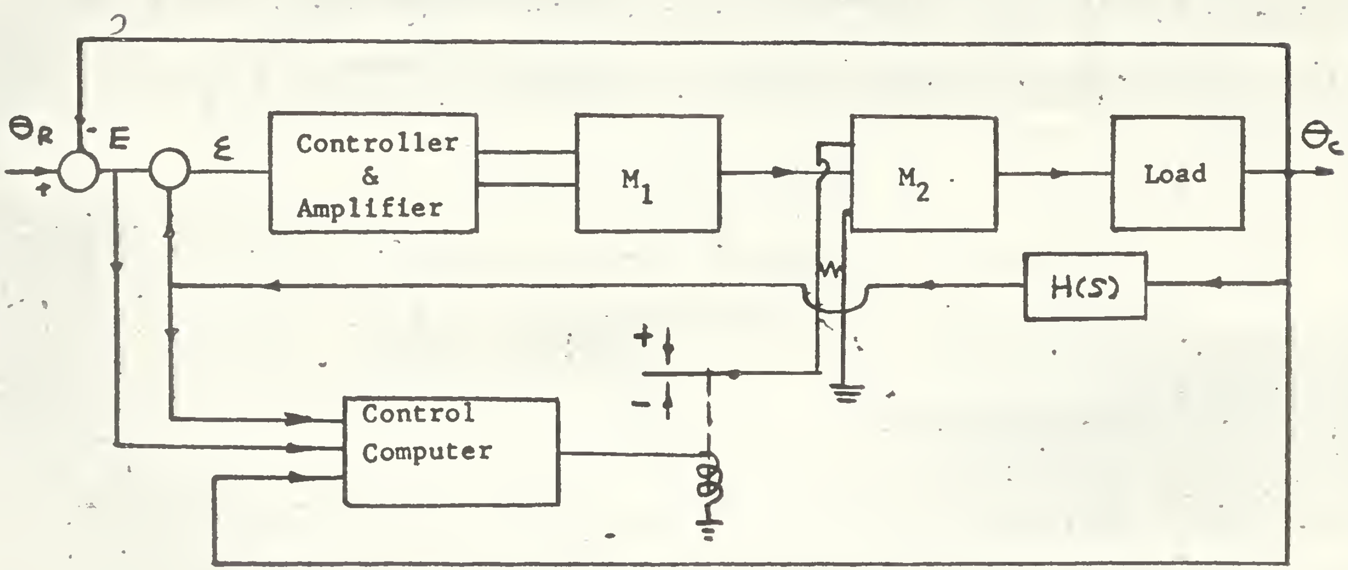


Fig. 6-15 Block Diagram of Continuous System with Relay Servo Accelerating and Braking

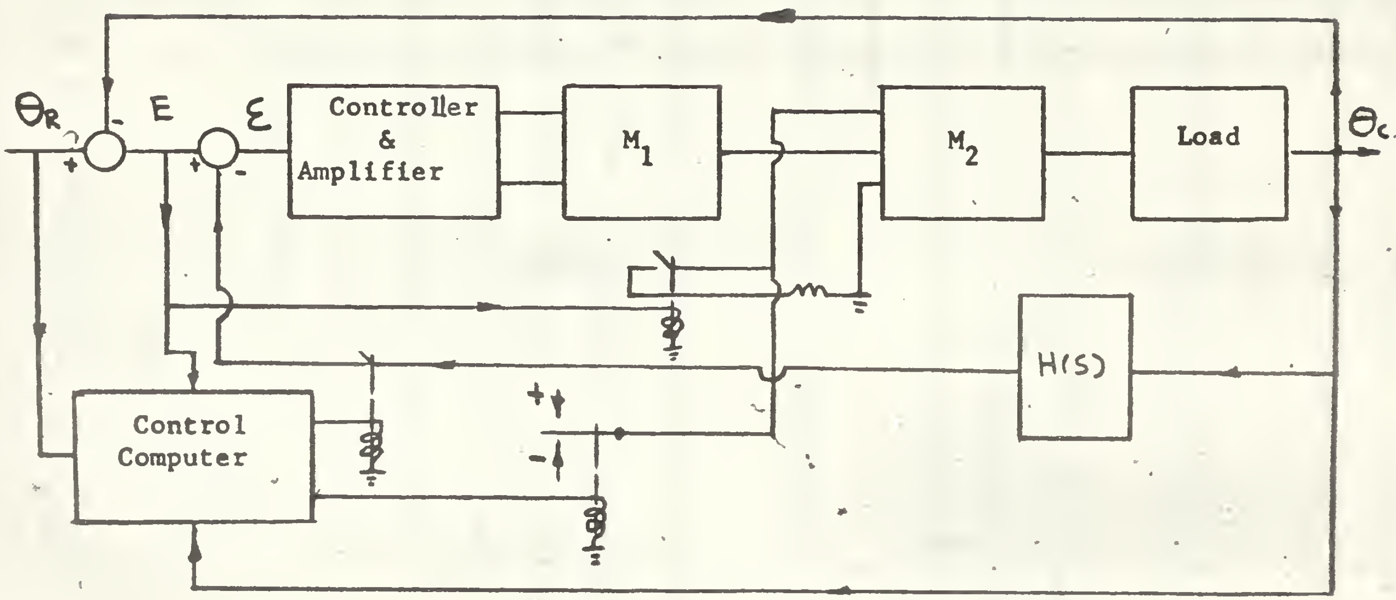


Fig. 6-16 Block Diagram of Discontinuous System with Controlled Feedback and relay Servo Braking.

in the accelerated part of the trajectory, therefore this is better than the electric damper as mentioned before. The die-out trajectory may be a straight line or with little curvature. The instant for accelerating and braking is controlled by the computer. A switch may be used to control the braking resistor R. The typical response curve for step inputs will be like that in Fig. 6-14.

Case 3: <u>Discontinuous system with controlled feedback and relay servo braking.</u>

From Fig. 6-16 the difference between this system and the other two systems is that the feedback signals are also controlled by the control computer. Based on the previous discussions the analysis about the character of this system is not necessary.

It may be helpful to analyze a simple case by using the phase plane method:

By assuming the time constants of both motors are negligible and the torque and braking applied by M_2 are equal to constants T and F_2, respectively, then the general equation will be $J\ddot{\theta}_c + F\dot{\theta}_c = KK_1E + T$

where K and K_1 are the gain of main amplifier and the first motor respectively. For accelerating operation with step input (Model 1).

$$J\ddot{E} + F\dot{E} + KK_1E = -T$$

or

$$\ddot{E} + A\dot{E} + BE = -C$$

where $A = F/J$, $\quad B = KK_1/J$, $\quad C = T/J$

Let $N = d\dot{E}/dE$ then

$$N = \frac{-C - A\dot{E} - BE}{\dot{E}}, \qquad \dot{E} = \frac{-(C+BE)}{A+N}$$

For $N = 0$, $\dot{E} = \dfrac{-(C+BE)}{A}$

Therefore, the larger the value of C the larger (more negative) the value of $\dot{E}$ for $N = 0$, this means the faster the response character. For the die-out part of the trajectory (Mode 2).

$$J\ddot{E} + (F + K_1 K K_T)\dot{E} + K K_1 E + F_2 \dot{E} = 0$$

Where K_T is the coefficient of tachometer feedback.

Then $\ddot{E} + A'E + BE = 0$ where $A' = F + F_2 + K_1 K_T K$

or $(S + \gamma_1)(S + \gamma_2) = 0$ where $\gamma_2 >>> \gamma_1$

after taking off the braking resistor (Model 3),

$$J\ddot{E} + (F + K_1 K K_T)E + K K_1 E = 0$$

$\ddot{E} + A''\dot{E} + BE = 0$ where $A'' = F + K_1 K_T K$

or $(S + \gamma_1')(S + \gamma_2') = 0$ where $\gamma_2' > \gamma_1'$

This is an overdamped case.

Since this three mode system cannot be analyzed only with discussions, therefore, a summary of investigation may be useful for the readers who like to make further investigations:

1. Complete general mathematical equations for three mode systems with high order motor load combinations.
2. How to decide the character of each motor according to a defined requirement.
3. Compare the characters of "relay", "linear" and "discontinuous" type of three mode systems.
4. The effect of motor time constants.
5. The application for A. C. systems.
6. Ramp, step and frequency response of some specific (example) system.
7. The load character effect.
8. Saturation and other non-linear effects.

CHAPTER VII - DIVIDED PHASE SPACE THEORY APPLIED TO DISCONTINUOUS SYSTEMS.

SECTION I - GENERAL INTRODUCTION

In the conventional Bang-Bang system, the switching devices are instrumented to follow the die-out trajectory. In cases when the die-out trajectory cannot be a straight line, then a parabola switching curve is best. Because this is hart to instrument, approximation with several straight line segments is commonly used.

In the discontinuous system, if the system is designed to have underdamped and overdamped characters that are interchangable by a simple relay action, then this kind of system will be better for step input than the corresponding optimum switched system, also better than the continuous system.

One theory presented before is to use the underdamped character to give a fast rising trajectory and the overdamped character to brake the system in the die-out part of the trajectory. The die-out trajectory is a straight line or chattered form which stays inside the linear zone. This has been proved good for linear systems with step input. By using a translated or modified switching method nearly optimum character is obtained for ramp inputs also. Since the overdamped character exists inside the relay dead zone only, therefore the whole phase plane or phase space is full of underdamped trajectories. If the input signal is a random wave form, and if the changing period of the input wave is quite fast, then in some cases the die-out trajectory will not be able to reach the origin of the phase plane (or error space). The apparent behaviour of this system will be an underdamped system. If the relay dead zone is too narrow, then it is very easy to make the relay reoperate due to nonlinearity. The idea of using a switching line with slightly less negative slope than that of the fast eigenvector was suggested, but

the system response character will still be nearly like an underdamped system if subjected to fast random inputs. If the relay dead zone is too wide, the response character for small input signals will be slow, and the trajectories for large input signals may have undershoot or overshoot due to the drop out line of the relay which does not pass through the origin of the phase plane. These little tails on the transient response will also be very slow as indicated in Fig. 2-16. On the other hand, the trajectories in quadrant 1 and 3 are underdamped, if a trajectory starts in these two quadrants, it will at first enlarge the error and then curve back, this will be slower than the continuous system. Finally; the discontinuous system, by using a heavily damped character in the relay dead zone, can have nearly optimum character only for step and ramp input, for fast and large random input signals the system is like an underdamped continuous system.

In order to design a system to be better than the continuous system for random input, the divided phase space or phase plane theory will be discussed in this section.

A phase plane can be divided into four parts as in Fig. 7-1. The dividing lines are the fast eigenvector and the E axis. In part II and VI, of the phase plane, the system is in an underdamped condition as used before; but part I and III are in a heavily damped condition, the trajectories tend to go to the E axis vertically, they will not enlarge the error very much as the underdamped trajectories do.

Since the concept of designing a system to have a fast response character is to make the trajectory in the phase plane or phase space go to the origin in a minimum amount of time, in other words in phase plane, for same length of distance on E axis, the trajectory which covers a larger area will be the faster one. Therefore, if the system has different characters according to

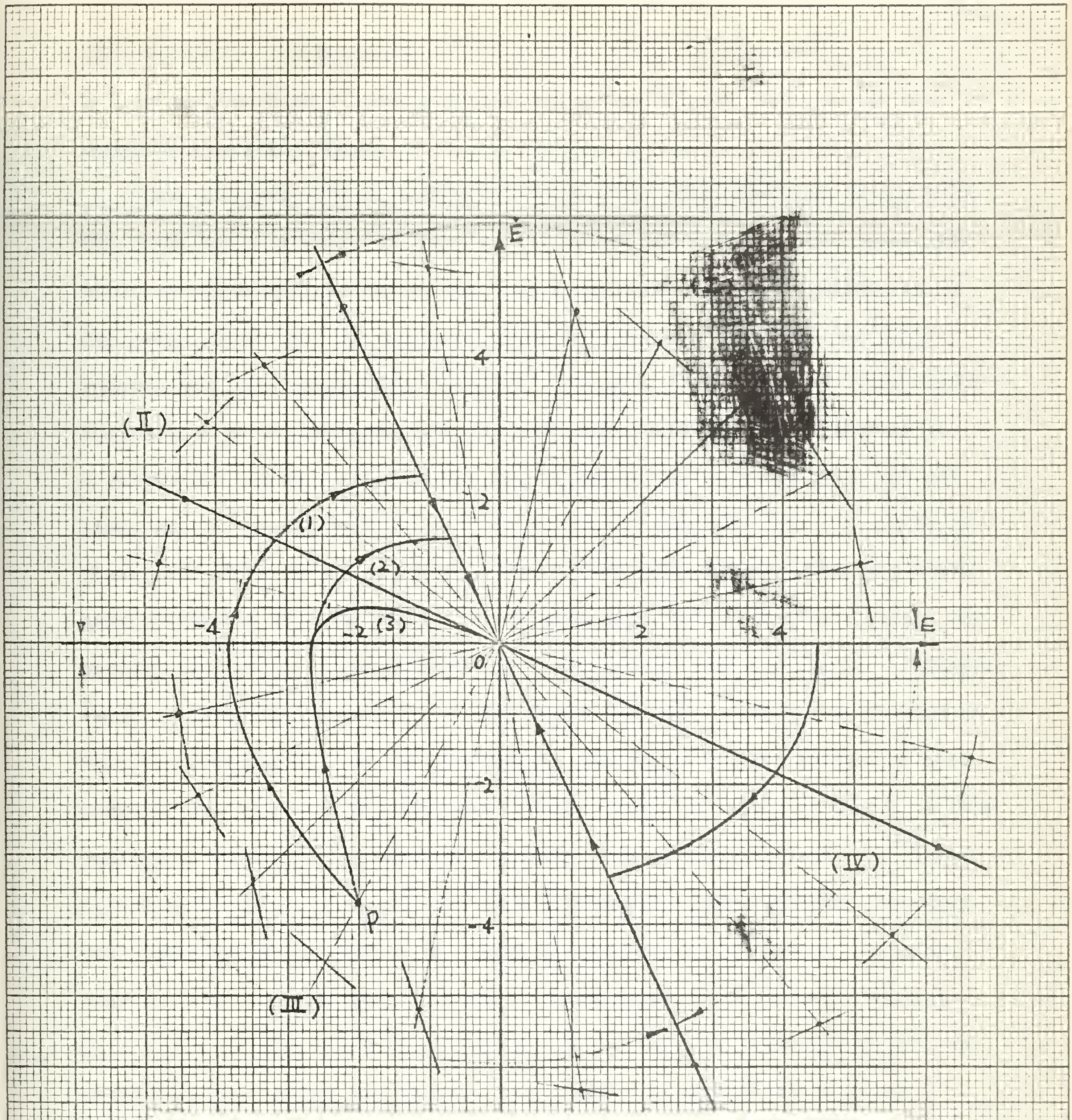


Fig. 7-1 Divided Phase Plane Plot for Step Input and Initial Conditions.

(1) Under damped (Continuous)
(2) Discontinuous
(3) Over Damped (Continuous)

} Trajectories

(I) and (III) Over Damped Parts

(II) and (IV) Under Damped Parts.

the position of the state point in the phase plane as indicated in Fig. 7-1, it will be better than the continuous system for any kind of initial conditions or step inputs.

This statement may be extended to high order phase space by syaing this The phase space (stationary error space) can be divided into four parts, the limits of these four parts are the E vs $\ddot{E}$ plane and the hypersurface, as in the phase plane, part II and VI are underdamped, part I and III are heavily damped. The trajectory which starts at any point in this divided error space will have a faster response character than the corresponding trajectory of a continuous system. But as point out in Chapter II the high order derivative signals are too noisy to be used as control signals, then the limits of these four parts can be regarded as the E vs $\ddot{E}$ plane and the switching plane used in Chapter II, that is, the plane perpendicular to the E vs $\dot{E}$ plane, and passing through the optimum switching line in space for step inputs. Theoretically this is only optimum for a step input, but the response for any kind of initial condition will be better than the continuous system, because in most parts of the space the discontinuous system gives fast response except in that small part between the hypersurface and the linear switching surface. Since these two surfaces are nearly combined for heavily damped systems, therefore the linear switching surface will give a very good approximation. Even in case the trajectories cause overshoot, the response is still fast, because once they pass over the E vs $\ddot{E}$ plane the system becomes underdamped, and the trajectory goes to the origin very fast. There is no undershoot in this error space, because the system gain can be raised to a high value, the dead zone of the relay can be neglected, and once trajectories cross over the switching surface they can never come back due to the underdamped character in part II and VI of the error space.

Because the switching surface in this divided error space is very easy to control and the trajectories are continuous except at the limiting surfaces, therefore the effect of nonlinearity will not be so critical as in the conventional optimum relay servo systems. The changing of the character of the system due to different environment will not cause serious trouble. In a word, this kind of system will be better than the continuous system as long as the different parts of the error space are at under and overdamped condition as mentioned before, and the only thing needed is to add some amplifiers and polarity sensitive relays, or a simple switching computer.

SECTION II - TRANSLATED AND DIVIDED PHASE PLANE AND PHASE SPACE THEORY APPLIED TO RAMP INPUT ANALYSIS.

As discussed in Chapter V, the ramp input tends to translate the phase plane or phase space along the positive or negative E axis. The theory of using a translated eigenvector to switch the trajectory onto the underdamped stable point of the ramp input has been proved to be practical both by phase plane plot and by computer results.

In Fig. 4-2, the isoclines indicated that if the trajectory starts from a stated point under the relay dead zone, then it will spiral around the underdamped stable point U, because any place outside the relay dead zone is in an underdamped condition. If the stable condition is to be obtained as early as possible, then heavy damping must be applied in the region under the relay dead zone since all the heavily damped trajectories are nearly vertical lines in this region. On the other side of the relay dead zone, all the underdamped trajectories are turning back to the underdamped stable points; but the overdamped trajectories are moving away from this point, therefore it is necessary to apply an underdamped condition in the region above the relay dead zone.

This statement may be extended by using the phase space concept that the translated hypersurface divides the whole phase space into two half-spaces the upper half-space is in underdamped condition and the lower half-space is in overdamped condition. Since the gain of the relay can be designed very high, the dead zone of the relay can be neglected. The relay circuit may be designed to have a nearly vertical switching line for the ramp input to avoid the chattering of the relay and to let the trajectories reach the minimum lagging error condition earlier.

One thing to consider here is the opposite polarity of the ramp input. As the computer result for a second order discontinuous feedback system in

Fig. 7-2 indicates that an overdamped condition is needed in the region between these two translated switching lines, and underdamped or overdamped condition outside this region. Therefore, the whole phase plane or phase space is divided into three parts as indicated in Fig. 7-2.

For a third order system translated hyperplane switching may be used or a plane passing through the stable point U and perpendicular to the E vs $\dot{E}$ plane. The slope of this plane may be adjusted to give optimum response to ramp input. The overshoot caused by different initial conditions can be eliminated by another switching operation. In any event for the same starting point in phase space the trajectory for a discontinuous system will be faster than that for the corresponding continuous system.

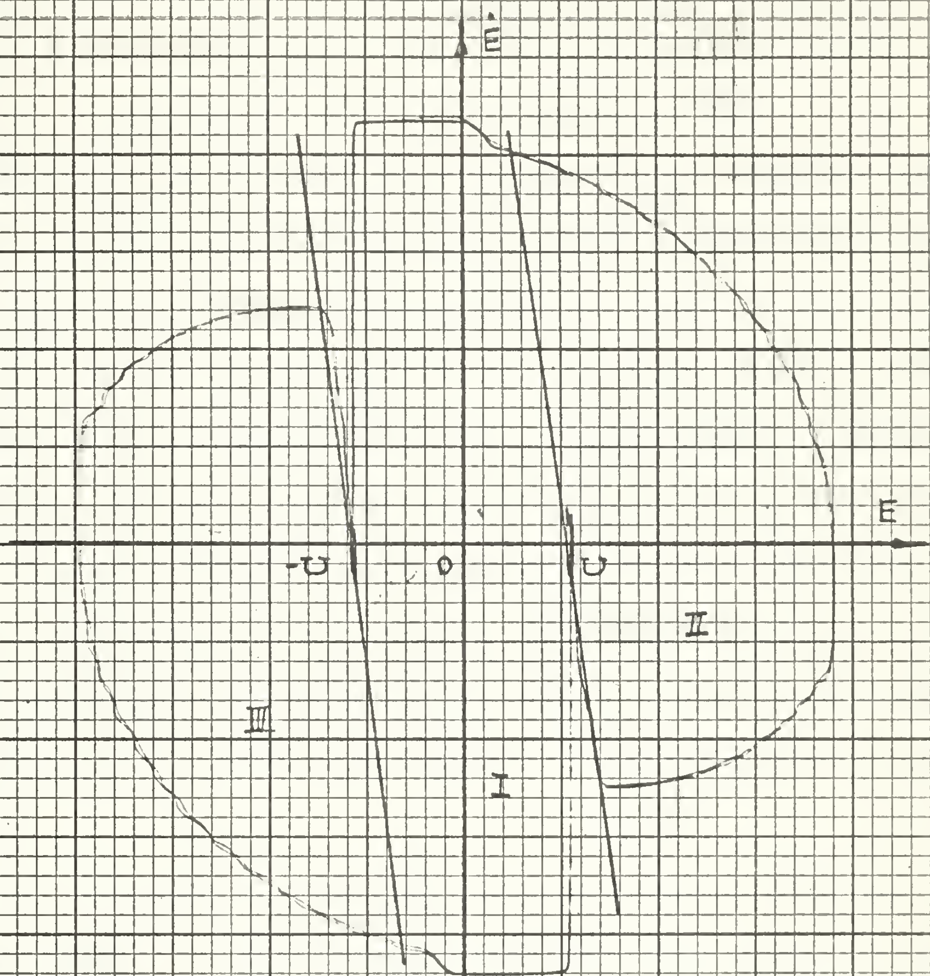


Fig. 7-2 Ramp response trajectory for both polarity of input (2nd order discontinuous system)

(I) Always over damped.
(II) Underdamped for positive ω_i over damped for negative ω_i.
(III) Over damped for positive ω_i Under damped for negative ω_i.

SECTION III - DESIGN CONSIDERATIONS

A. Step input:

From Fig. 7-1, for a second order system with step input or stationary initial conditions (no derivative signal input), the switching actions are needed both on the E axis and the fast eigenvector, and both of them are polarity sensitive. A simple relay circuit is suggested in Fig. 7-3, in which the $\dot{E}$ signal is used as a control signal for relay R_1. When the input signal is a positive step or the trajectory starts in the upper half of the phase plane the relay R_2 is directly connected to the output of the control signal amplifier μ, as the trajectory hits the fast eigenvector, R_2 closes the feedback path, and also keeps closed when the control signal is changed to opposite polarity. If the trajectory overshoots the relay R_1 will reverse the polarity of the control signal just as the trajectory hits the E axis. Since R_2 is also polarity sensitive, therefore the feedback path will be opened due to the reoperation of the relay R_2, and the system becomes underdamped again, then there will be no slow response at the end of the trajectory.

B. Ramp input:

From Fig. 7-2, the method for dividing the phase plane is different from that for step inputs. Here the dividing lines are these two translated eigenvectors. Their equation is:

$$r\left(E - \frac{a\omega_i}{K}\right) + \dot{E} = 0 \qquad (7\text{-}1)$$

When ω_i is positive, E is positive, $\dot{E}$ changes from positive to negative. The sign of μ is positive in the upper side of the translated eigenvector in quadrant one, two and four. Then the whole phase plane can be divided into two parts by this polarity sensitive relay action. In case the ramp input is negative, E is negative, $\dot{E}$ changes from negative to positive, the sign of is changing from negative to positive, but an underdamped

condition is needed. When the sign of u is negative, therefore if the same relay is used as for positive ramp input another relay is needed to change the sign of the output of the control amplifier. This is the same switching circuit as for step input except another signal $\frac{-a\omega_i}{K}$ is added to the control amplifier and the control signal $\dot{E}$ for R_1 is replaced by as indicated in Fig. 7-3.

Since the switching circuit is designed to identify the character of input signal and initial conditions, then this system will be self-adaptive to the input signal and initial conditions.

For third or higher order systems add another derivative signal to the control amplifier to use hypersurface switching, or use this same circuit, because the ramp input effect is the same for any order system. Since the system has no under shoot and no slow response in the over shoot part of the trajectory, then the exact design of the switching hypersurface is not required.

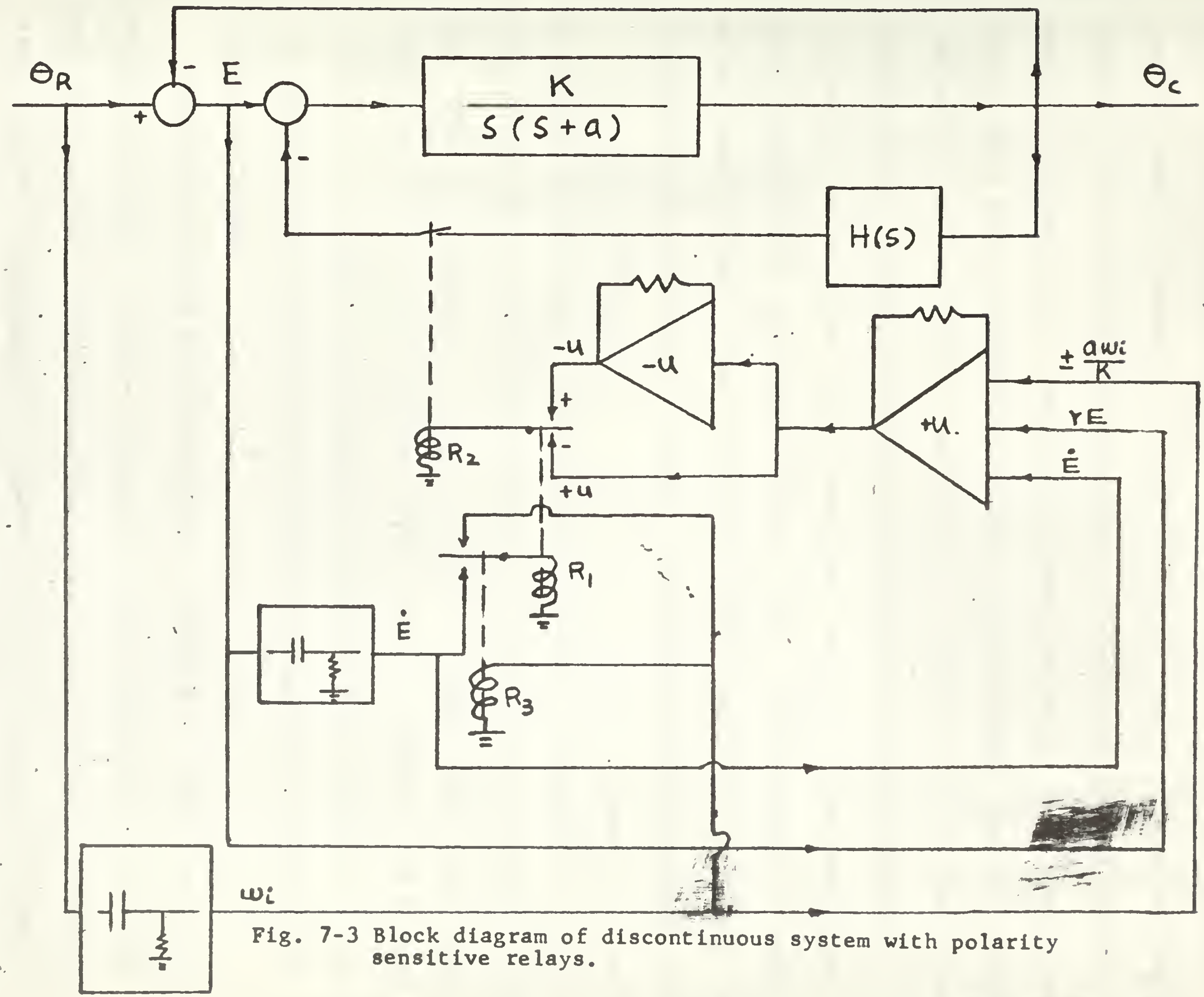


Fig. 7-3 Block diagram of discontinuous system with polarity sensitive relays.

SECTION IV - RELAY SERVO OR BANG-BANG SYSTEM DESIGN CONSIDERATION

The die-out part of the relay servo system, if it can be made overdamped, such as by using dynamic braking and various dampers, as in the previous sections, then the same kind of theory can be applied to improve the response character. The only change is to control the damping or braking circuit instead of a feedback path. The divided phase plane plot will be nearly the same as for a linear second order feedback control system, because a braking condition is used in the part I and III as in Fig. 7-1, and all the trajectories are nearly straight lines. When considering the response character for all the input signals and initial conditions, any part of the phase plane can be regarded as a starting or a die-out part. The improvement is based on the assumption that in any part of the phase plane the discontinuous trajectories can be made better than that of the corresponding continuous system.

If dynamic braking is used to damp the relay servo system as in Fig. 7-4, the braking resistor should be controlled by a switching computer, and the power supply path to the motor should be controlled by the switching computer also. That is, whenever R_2 closes R_3 must be opened. The stored energy braking method may also be applied in this case the analysis will be more complex due to the energy storage parameter caused by the inductor, but the result may be better than that of the dynamic braking case provided the adjustment is proper.

Three mode systems were investigated in Chapter VI, there the die-out trajectory is made a straight line near the fast eigenvector in the relay dead zone, but here the braking effect is used in all parts of the phase plane below the fast eigenvector, therefore the switching circuit should be polarity sensitive. The other modification can be derived from the investigations made in this chapter.

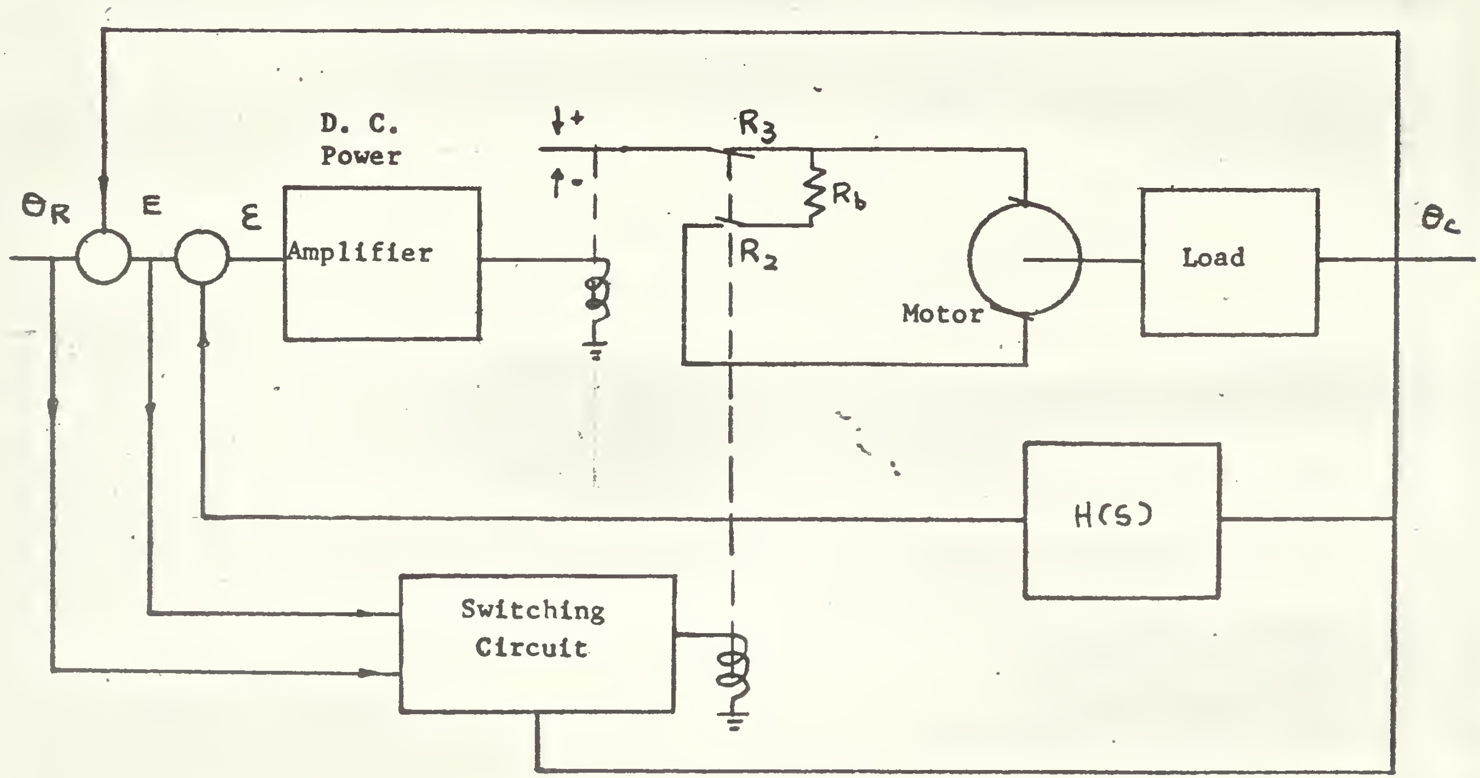


Fig. 7-4 Relay servo with controlled dynamic braking resistor.

For saturated type bang-bang systems one thing of interest is the improvement of the ramp input. By using the discontinuous braking method the result may be better than that in Chapter V, since the trajectories under the switching line are improved. For high order systems the approximation method will be just like the divided phase space concept given before.

SECTION V - TRANSLATED EIGENVECTOR AND HYPERPLANE SWITCHING APPLIED TO TYPE TWO SYSTEMS WITH ACCELERATION INPUT AND DISCONTINUOUS FEEDBACK COMPENSATION.

So far the optimum switching methods discussed are only concerned with type one systems. But in fact this phase plane and phase space theory can also be used to design optimum or nearly optimum discontinuous systems with type number higher than one. In this section a brief discussion of switching lines and surfaces for type two 2nd and 3rd order systems will be given.

A. 2nd Order Type Two System with Discontinuous Feedback Compensation.

From Fig. 7-5, the equation of the system is

$$\ddot{\theta}_c = K(E - K_T\,\dot{\theta}_c) \tag{7-2}$$

i.e.

$$\ddot{E} - K K_T \dot{E} + K E = \ddot{\theta}_R + K K_T \dot{\theta}_R \tag{7-3}$$

For $\theta_R = \alpha_i t^2$, then $\dot{\theta}_R = 2\alpha_i t$, $\ddot{\theta}_R = 2\alpha_i$, $\dddot{\theta}_R = 0$

and the characteristic equation is

$$\ddot{E} + K K_T \dot{E} + K \left[E - \frac{2\alpha_i (1 + K K_T t)}{K} \right] = 0 \tag{7-4}$$

Let $N = \dfrac{d\dot{E}/\omega_n}{dE}$

then the isocline equation is

$$\frac{\dot{E}}{\omega_n} = - \frac{E - \frac{2\alpha_i}{K}(1 + K K_T t)}{N + K_T} \tag{7-5}$$

For $K_T = 0$,

$$\frac{\dot{E}}{\omega_n} = \frac{-E + 2\alpha_i/K}{N} \tag{7-6}$$

Eq. (7-4) shows that the tachometer feedback changes the system to type one. But here this feedback signal is used to damp the system and to drive the overdamped trajectory through the underdamped stable point. When the stable point is reached this feedback path will open. Then the system will again be a type two system, and follow the acceleration input signal with a definite minimum lagging error of $\frac{2\alpha_i}{K}$ as indicated in Eq. (7-6). In other words the improvement is accomplished by changing the type

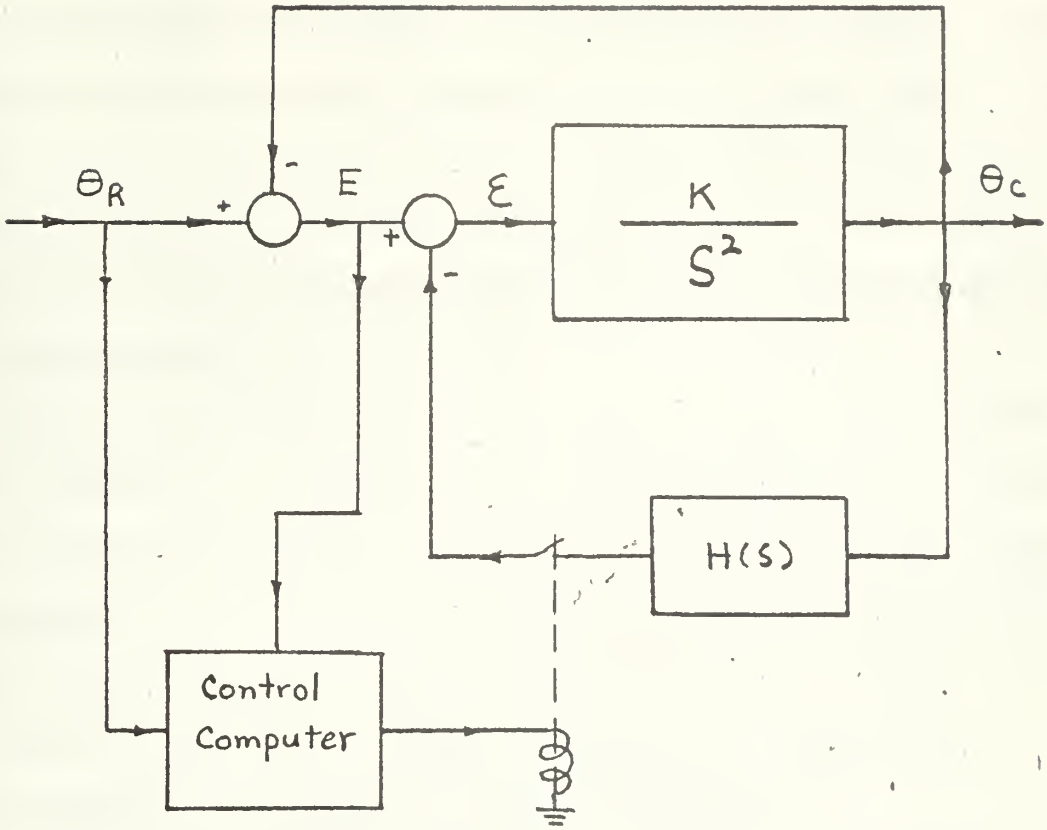


Fig. 7-5 Block Diagram of a 2nd Order Type Two System With Discontinuous Feedback.

of the system and its damping at a proper instant, and the particular character of type two system for following an acceleration input can be kept by switching off the feedback signal.

The isocline equation for $K_T = 0$ is the equation for a family of circles with their center at $E = \frac{2\alpha_i}{K}$ (for positive acceleration input). If there is no continuous feedback the trajectory for an acceleration input will never reach a stable point, since there is no damping. On the other hand if the system is switched as a conventional Bang-Bang system, the switching computer will be very complex even if restricted to optimum response of step inputs or stationary initial conditions.

But in this discontinuous system there is a time variable in the isocline equation due to the tachometer feedback. Geometrically this corresponds to the overdamped focal point moving in the positive error direction as time increases. This is the character of a type one system. The following numerical example will illustrate this effect.

Let $K = \omega_n^2 = 64$, $\alpha_i = 1$ rad/sec^2, $K_T = 0.341$

The underdamped trajectory is a circle with radius of

$$R = \left[\left(\frac{\alpha_i}{\omega_n}\right)^2 + \left(\frac{2\alpha_i}{K}\right)^2\right]^{\frac{1}{2}} = \left[(0.125)^2 + (0.0313)^2\right]^{\frac{1}{2}} = 0.127$$

The overdamped isocline equation is

$$\frac{\dot{E}}{\omega_n} = \frac{\frac{2\alpha_i}{K}(1 + KK_T t) - E}{N + K_T} = \frac{0.0313(1 + 21.8t)}{N + 0.341}$$

Consider that the time when the underdamped trajectory starts from $\dot{E}/\omega_n$ axis till it hits the translated fast eigenvector is 1/2 t_o (where $t_o = \frac{1}{\omega_n} = \frac{1}{8}$ sec.), and the time when the overdamped trajectory hits the fast eigenvector is t_o, then two sets of isoclines can be constructed and an approximate trajectory can be plotted as in Fig. 7-6. In this case, the value of t is very small at the first switching instant, the overdamped trajectory is not much different from the underdamped trajectory at the beginning, then there will be a second switching operation at point P_2 in Fig. 7-6, and a third

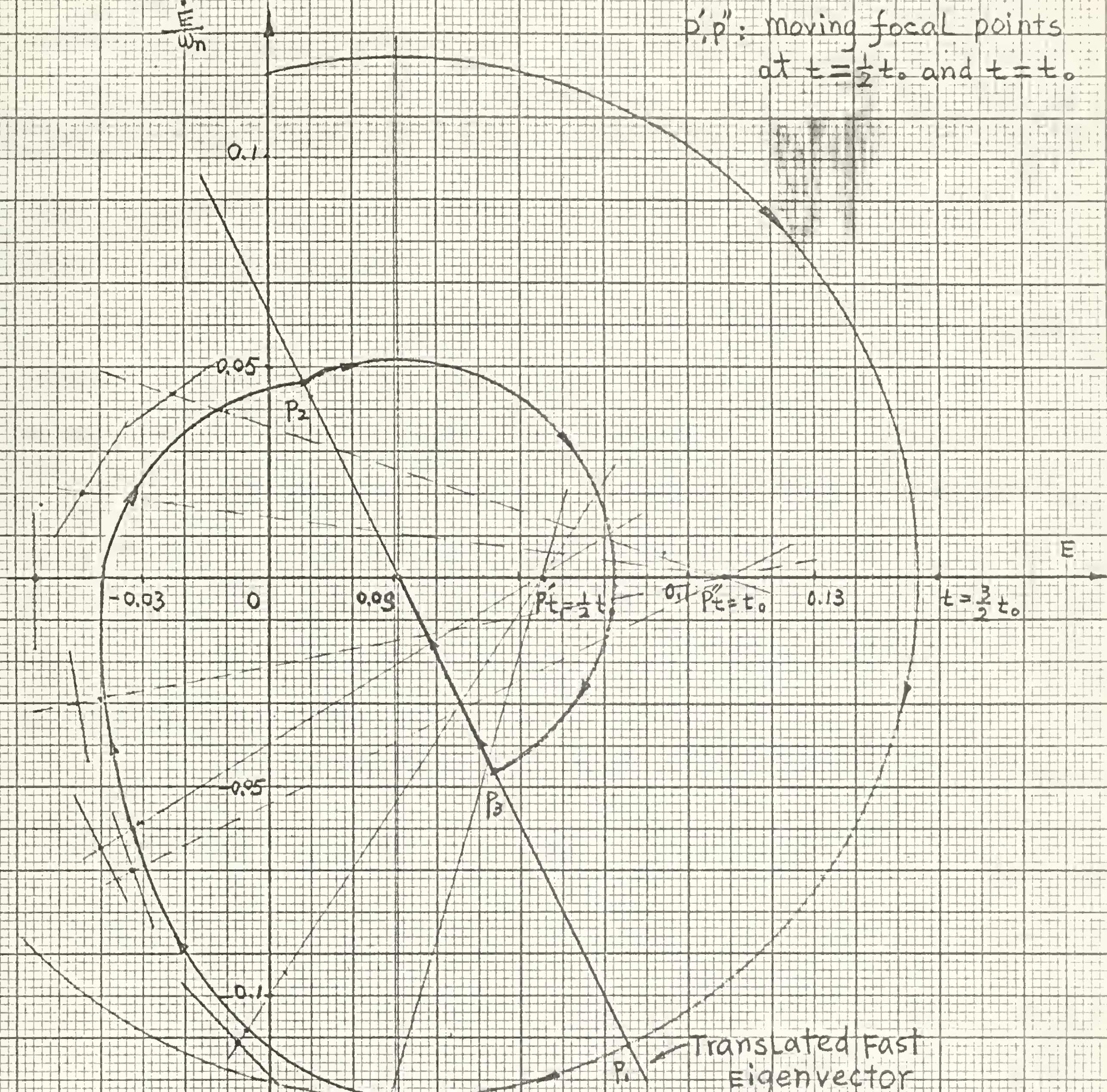


Fig. 7-6 Phase Plane Plot for a 2nd Order Type Two System with Discontinuous Feedback Compensation and Acceleration Input.

switch operation at p_3. Since the overdamped stable point is far away after this third switching operation, the trajectory will follow the fast eigenvector and go to the underdamped stable point.

The other switching line may also be used to apply the feedback damping earlier and make the overdamped trajectory just pass through the underdamped stable point and then switch out the damping signal, but the switching computer may be very complex.

On the other hand, if continuous tachometer feedback is used to compensate this kind of system, the lagging error will be very large as time increases even if the feedback coefficient is small.

B. 3rd Order Type Two System with Discontinuous Feedback Compensation.

From Fig. 7-7, the equation of this system is

$$\dddot{\theta}_c + a\ddot{\theta}_c = K[E - K_a\ddot{\theta}_c - K_T\dot{\theta}_c]$$

i.e.

$$\dddot{E} + (a+KK_a)\ddot{E} + KK_T\dot{E} + KE = (a+KK_a)\ddot{\theta}_R + KK_T\dot{\theta}_R \tag{7-7}$$

The characteristic equation is

$$\dddot{E} + (a+KK_a)\ddot{E} + KK_T\dot{E} + K\left\{E - \frac{(a+KK_a)\ddot{\theta}_R + KK_T\dot{\theta}_R}{K}\right\} = 0 \tag{7-8}$$

For $K_a = K_T = 0$

$$\dddot{E} + a\ddot{E} + K\left(E - \frac{a\ddot{\theta}_R}{K}\right) = 0 \tag{7-9}$$

For $\theta_R = \omega_i t^2$, Eq. (7-8) and Eq. (7-9) become

$$\dddot{E} + (a+KK_a)\ddot{E} + KK_T\dot{E} + K\left\{E - \frac{2\alpha_i[(a+KK_a)+KK_Tt]}{K}\right\} \tag{7-10}$$

and

$$\dddot{E} + a\ddot{E} + K\left(E - \frac{2a\alpha_i}{K}\right) = 0 \tag{7-11}$$

From Eq. (7-10) and Eq. (7-11), the same statements for the above 2nd order type two system can be used here, but in this case the trajectories are in phase space instead of in the phase plane, and an additional lagging error is caused by the coefficient of acceleration feedback K_a, then for minimizing the final error both the tachometer and the acceleration feedback need

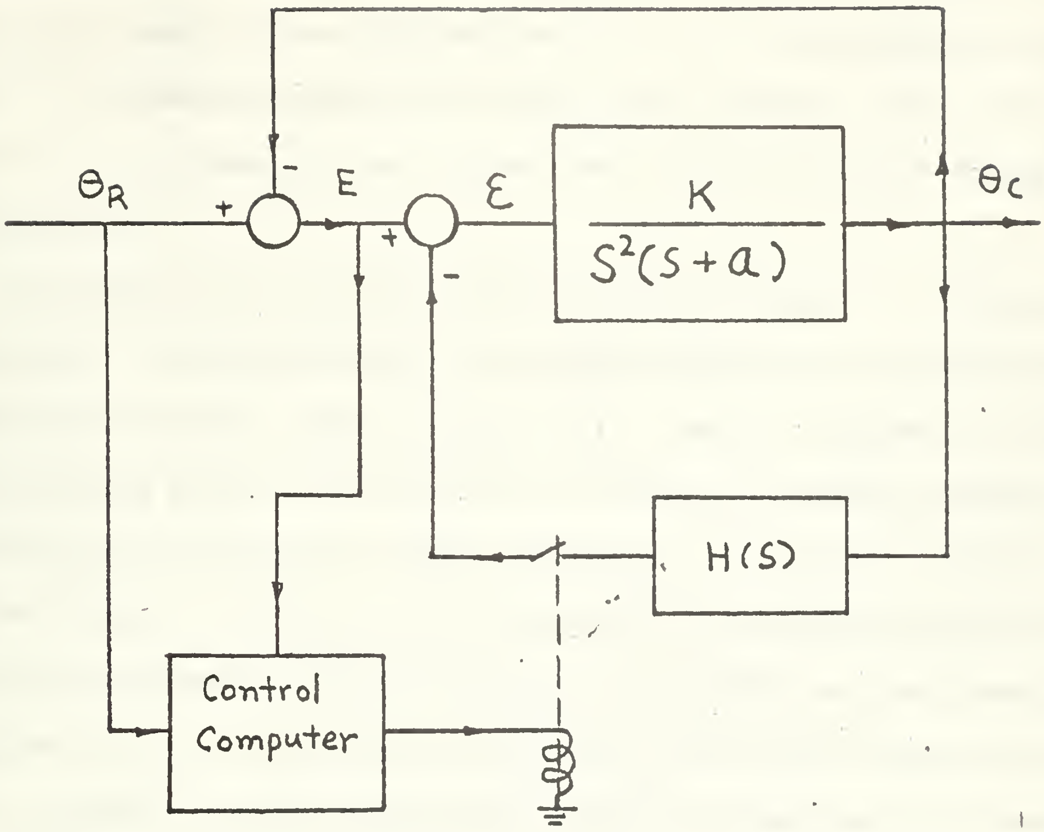


Fig. 7-7 Block Diagram of a 3rd Order Type Two System with Discontinuous Feedback.

to be switched out and the minimum lagging error for acceleration input is $\frac{2a\alpha_i}{K}$. When the acceleration feedback signal is switched in, the overdamped stable point will be moved away from the underdamped stable point $\mho$ even at t equal to zero, for small values of a the overdamped trajectories near the underdamped stable point will be nearly vertical (their projections onto the E vs $\dot{E}$ plane), therefore the translated hyperplane switching theory is applicable.

In summary, for a feedback control system with order n and type m, the effect of the mth derivative of the input signal is to translate the stationary error space along the error axis. Improved response for this mth derivative input can be obtained by translated hyperplane switching theory, to control the discontinuous feedback signals. The effect of the input signals with orders less than m is to enlarge the lagging error that is caused by the mth derivative signal (for negative feedbacks), and the effect is a function of time and only acts in the period when feedback signals are switched in. If the overdamped trajectories near the E axis are nearly vertical (the projections onto the E vs $\dot{E}$ plane), the translated hyperplane or the approximate switching plane (by using E and $\dot{E}$ as control signals) will be in a converged region in the error space. The same concept can be applied to Bang-Bang systems with damping devices. And the divided error space theory will also be useful for considering random input signals and initial conditions. Here the particular character of the discontinuous system is based on phase space methods to simplify the switching computer and to be self adaptive to various input signals and initial conditions.

SECTION VI - RAMP RESPONSE ANALYSIS OF 2ND ORDER SYSTEM WITH DISCONTINUOUS FEEDFORWARD COMPENSATION.

From Fig. 7-8

$$\ddot{\theta}_c + a\dot{\theta}_c = K[E + k_T\dot{E}] \tag{7-12}$$

Where K_T is the coefficient of $\dot{E}$ feedforward signal

$$\ddot{\theta}_c + a\dot{\theta}_c = KE + Kk_T\ddot{E} \tag{7-13}$$

$$\dot{\theta}_c = \dot{\theta}_R - \dot{E}$$

$$\ddot{E} + (a + Kk_T)\dot{E} + K\ E = a\dot{\theta}_R + \ddot{\theta}_R \tag{7-14}$$

For $\theta_R = \omega_i t, \quad \dot{\theta}_R = \omega_i, \quad \ddot{\theta}_R = 0$ then

$$\ddot{E} + (a - Kk_T)\dot{E} + K\ E = a\omega_i \tag{7-15}$$

Let $N = \dfrac{d\dot{E}/\omega_n}{dE}$ the isocline equation is

$$\frac{\dot{E}}{\omega_n} = \frac{\dfrac{a\omega_i}{K} - E}{N + \dfrac{a + Kk_T}{\omega_n}} \tag{7-16}$$

Let $a = 2.6, \quad K = 64 = \omega_n^2, \quad \omega_i = 1 \text{ rad/sec}, \; k_T = 0.3$

Then $$\frac{\dot{E}}{\omega_n} = \frac{0.0406 - E}{N + 0.325 + 2.4} = \frac{0.0406 - E}{N + 2.725} \tag{7-17}$$

For $k_T = 0$

$$\frac{\dot{E}}{\omega_n} = \frac{0.0406 - E}{N + 0.325} \tag{7-18}$$

The phase plane plot for this set of numerical values is given in Fig. 7-9. Equation (7-18) is the same isocline equation as used in Chapter V for Fig. 5-2, but Eq. (7-17) is different from Eq. (5-4) by an amount of in the numerator. Therefore the translation of the underdamped stable point is eliminated.

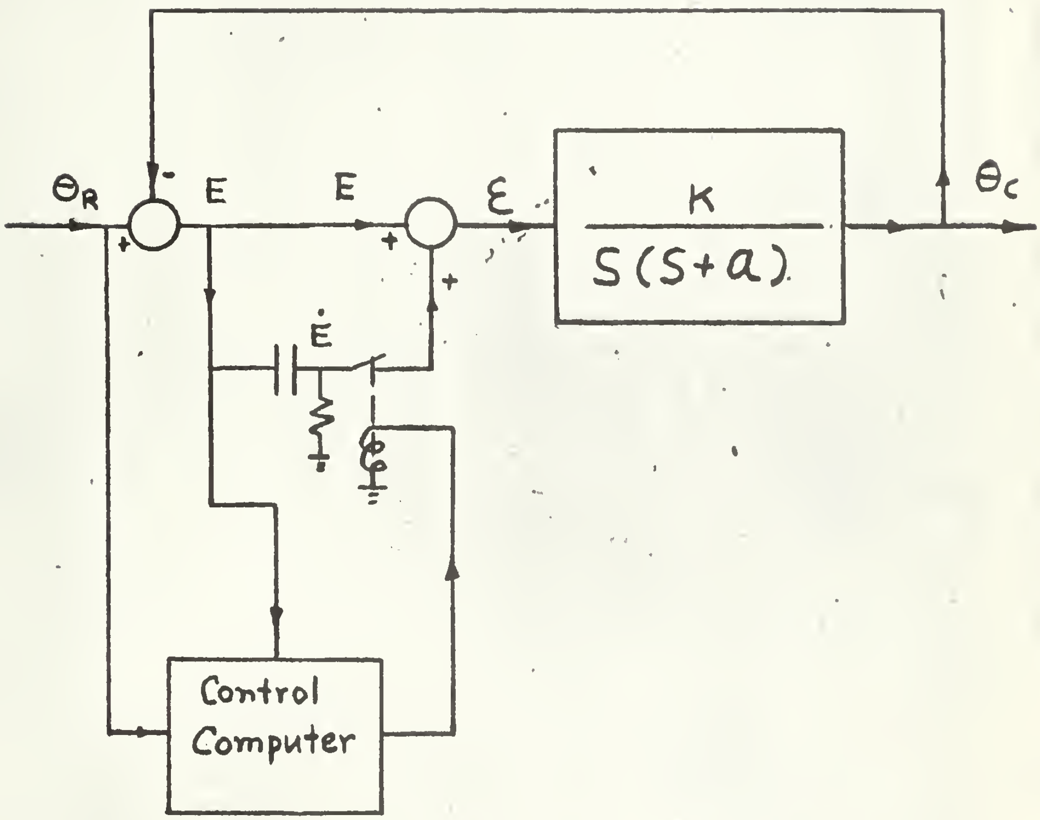


Fig. 7-8 Block Diagram of a 2nd Order System with Feedforward Compensation.

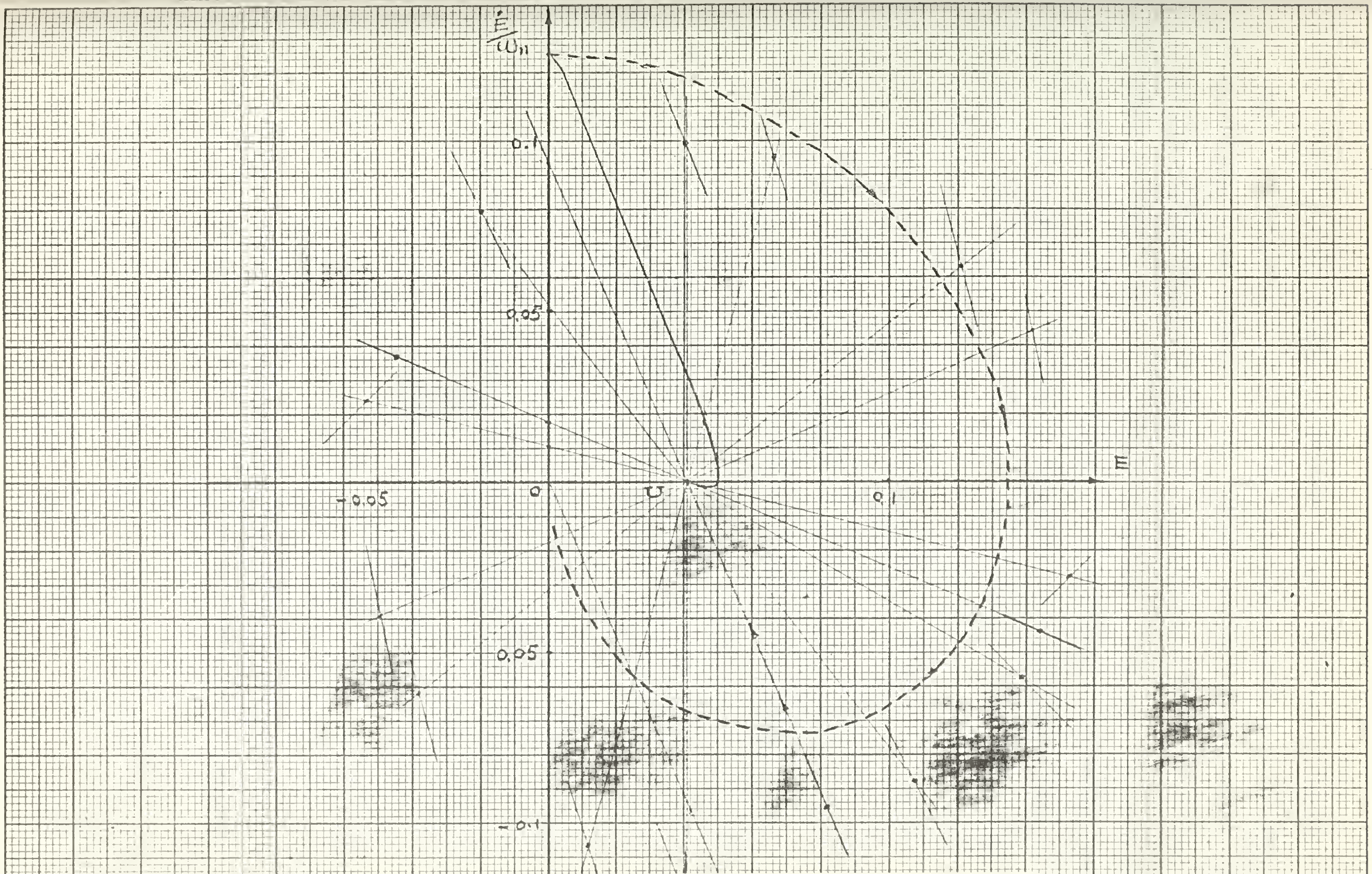


Fig. 7-9 Phase Plane Plot for a 2nd Order System with $\dot{E}$ Feedforward Compensation for Ramp Input.

In this phase plane plot, because the starting point of the trajectory is above the translated fast eigenvector, there is an overshoot at the end of the trajectory. The response will be very slow there. But if the value of k_T is too large, the starting point may be lower than the translated fast eigenvector, and an undershoot will occur. By using properly chosen values of k_T the optimum response character can be obtained.

The condition for making the translated fast eigenvector pass through the starting point of the ramp input on the $\dot{E}/\omega_n$ axis can be derived from the following equations:

Equation of the translated fast eigenvector is

$$\frac{\dot{E}}{\omega_n} = \frac{\frac{a\omega_i}{K} - E}{N + \frac{a + K k_T}{\omega_n}} \tag{7-19}$$

For $E = 0$, the intersection on the $\dot{E}/\omega_n$ axis has to be ω_i/ω_n, then

$$\frac{\frac{a\omega_i}{K}}{N + \frac{a + K k_T}{\omega_n}} = \frac{\omega_i}{\omega_n} \tag{7-20}$$

i.e.

$$k_T = \frac{-N}{\omega_n} \tag{7-21}$$

Since the slope of the translated fast eigenvector is N, then

$$N = -\frac{\frac{\omega_i}{\omega_n}}{\frac{a\omega_i}{K}} = -\frac{\omega_n}{a} \tag{7-22}$$

Substitute Eq. (7-22) in Eq. (7-23)

$$k_T = \frac{1}{a} \tag{7-23}$$

For the above example, $a = 2.6$, $k = 64$, $\omega_n = 8$, then $k_T = 0.385$, and

$r_1 = a = 2.6$, $r_2 = 24.6$

The isocline equation for the overdamped case is

$$\frac{\dot{E}}{\omega_n} = \frac{0.0406 - E}{N + 0.325 + 3.07} = \frac{0.0406 - E}{N + 3.395}$$

For the translated fast eigenvector, $N = -\frac{\gamma_2}{\omega_n} = -\frac{24.6}{8} = -3.07$

and for $E = 0$

$$\frac{\dot{E}}{\omega_n} = 0.125 = \frac{\omega_i}{\omega_n}, \qquad \omega_i = 0.125\,\omega_n = 1 \text{ rad./sec.}$$

The phase plane plot is in Fig. 7-10.

Because the trajectory in the phase plane exactly follows the fast eigenvector, therefore this is the optimum response for a ramp input. Since the value of k_T for this condition has no relation with the magnitude of the ramp input, as long as the linear relation holds, this is the optimum condition for any value of ramp input.

In order to consider the effect of nonlinearity and the random ramp initial conditions and inputs, the divided phase plane theory is applicable. But the underdamped and overdamped parts are divided just as for step inputs for a feedback compensated 2nd order system except the translation along the E axis as indicated in Fig. 7-10.

From this ramp input response analysis, the gain of the feedforward path has a definite relation to the open loop pole, but there is no relation between them for the step response, therefore the optimum design procedure needs to be started from the ramp input analysis rather than step input.

After the optimum response character for the ramp input is obtained this kind of system will naturally have optimum character for step inputs, because the control signal for shifting the hyperplane will become zero for step inputs.

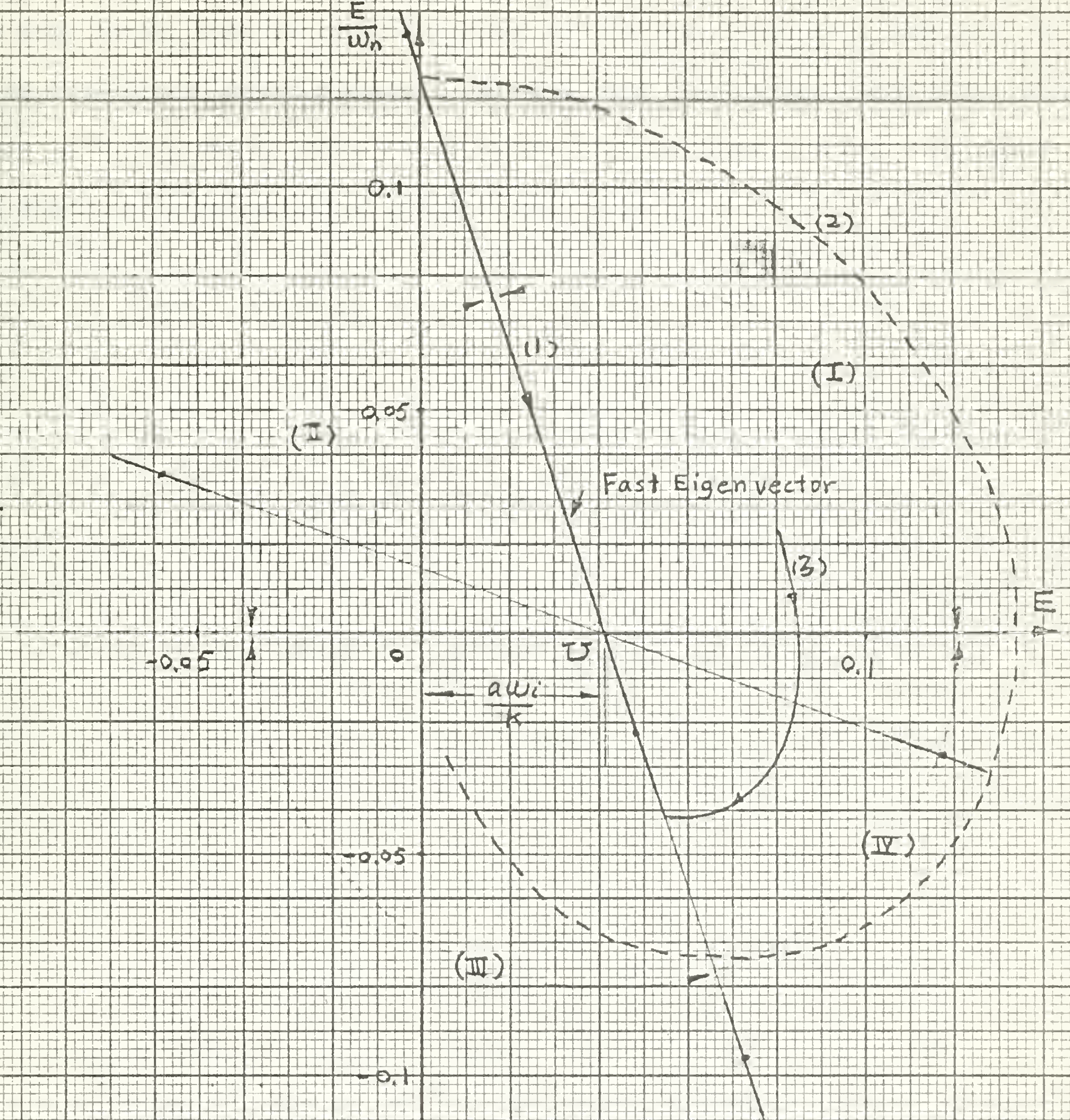


Fig. 7-10 Phase Plane Plot for a 2nd Order System with Discontinuous $\dot{E}$ Feedforward Compensation and Ramp Input.
(1) Optimum Trajectory.
(2) Underdamped trajectory
(3) Trajectory for an Initial Condition

Part (I) and (III) Overdamped
Part (II) and (IV) Underdamped

To improve the response of the system by using very heaving damping the fast eigenvcctor is nearly combined with the $\dot{E}$ axis of the phase plane for a step input, then the starting point of the ramp trajectory must stay below the translated fast eigenvector, in this case the divided phase plane theory has to be applied in order to eliminate the undershoot of the ramp input. A sketched illustration is given in Fig. 7-11.

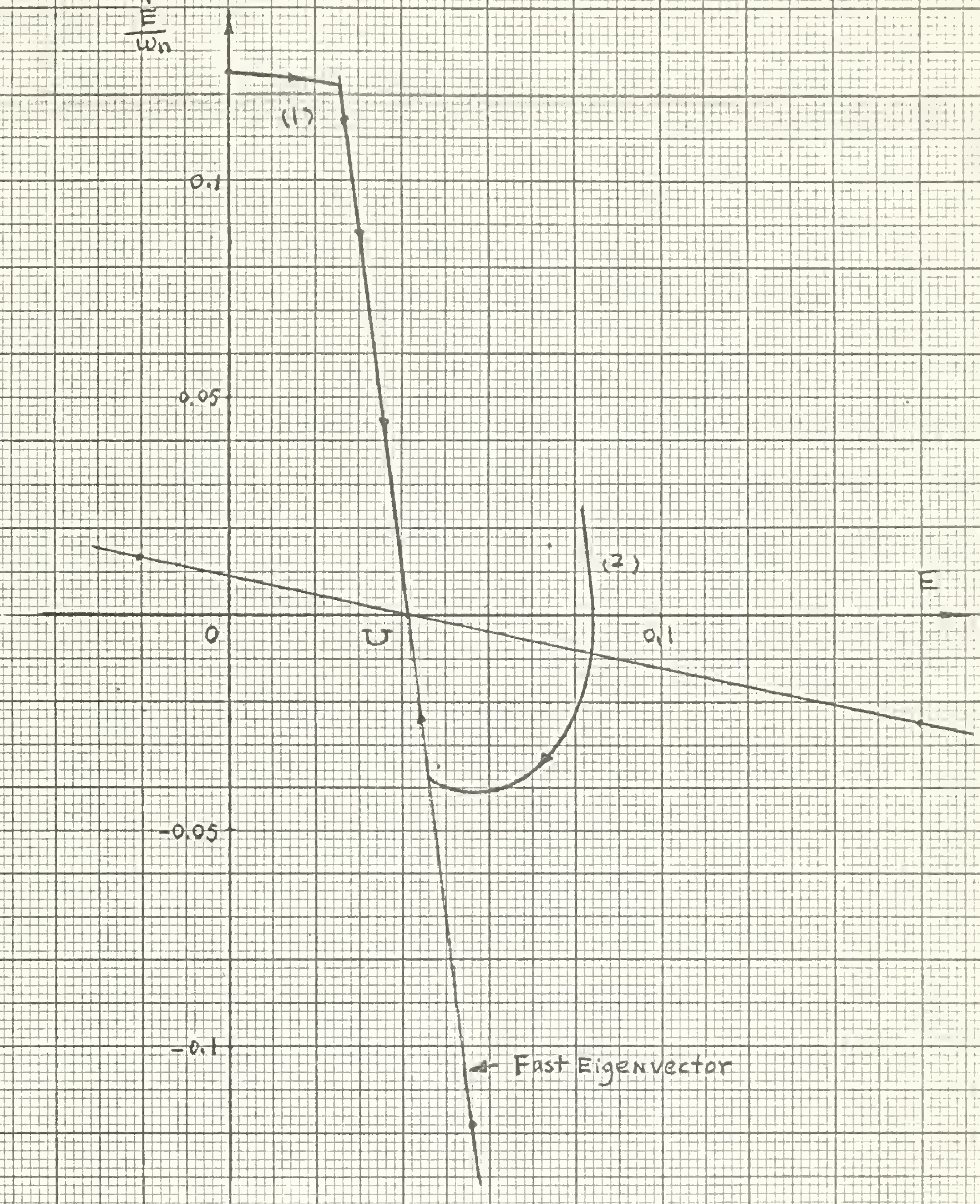


Fig. 7-11 Phase Plane Plot for a 2nd Order System with Large Discontinuous Feedforward Compensation and Ramp Input.
(1) Step Ramp Input Trajectory
(2) Trajectory of an Initial Condition.

SECTION VII - 3RD ORDER TYPE ONE SYSTEM WITH DISCONTINUOUS FEEDFORWARD AND FEEDBACK COMPENSATION.

From Fig. 7-12

$$\dddot{\theta}_c + A\ddot{\theta}_c + B\dot{\theta}_c = K\mathcal{E}$$

$$\mathcal{E} = E + K_T\dot{E} - K_a\ddot{\theta}_c$$

$$\ddot{\theta}_c = -\ddot{E} + \ddot{\theta}_R$$

$$-\dddot{E} + A(\ddot{\theta}_R - \ddot{E}) + B(\dot{\theta}_R - \dot{E}) = KE + Kk_T\dot{E} - KK_a\ddot{\theta}_R + KK_a\ddot{E}$$

$$\dddot{E} + (A + KK_a)\ddot{E} + (B + KK_T)\dot{E} + KE = \dddot{\theta}_R + (A + KK_a)\ddot{\theta}_R + B\dot{\theta}_R \qquad (7\text{-}24)$$

For a step input, $\ddot{\theta}_R = \dot{\theta}_R = 0$, Eq. (7-24) is a conventional 3rd order characteristic equation. Hyperplane switching methods can be applied. If the original uncompensated system cannot provide a fast rising character, the gains and the polarities of the feedforward and feedback signals may be controlled by the control computer.

For a ramp input, $\ddot{\theta}_R = 0$, $\dot{\theta}_R = \omega_i$, the minimum lagging error is $\frac{B\omega_i}{K}$ and translated hyperplane switching is applicable.

For an acceleration input $\theta_R = \alpha_i t^2$, $\dot{\theta}_R = 2\alpha_i t$, $\ddot{\theta}_R = 2\alpha_i$, $\dddot{\theta}_R = 0$, there will be a lagging error due to B and changing with time. Since this is a type one system, it cannot follow an acceleration input, but the stationary part of the lagging error can be reduced by feeding forward the $\ddot{E}$ signal to replace the $\ddot{\theta}_c$ feedback signal. If the value of B is not large, the system will approximately follow the acceleration input for a short period.

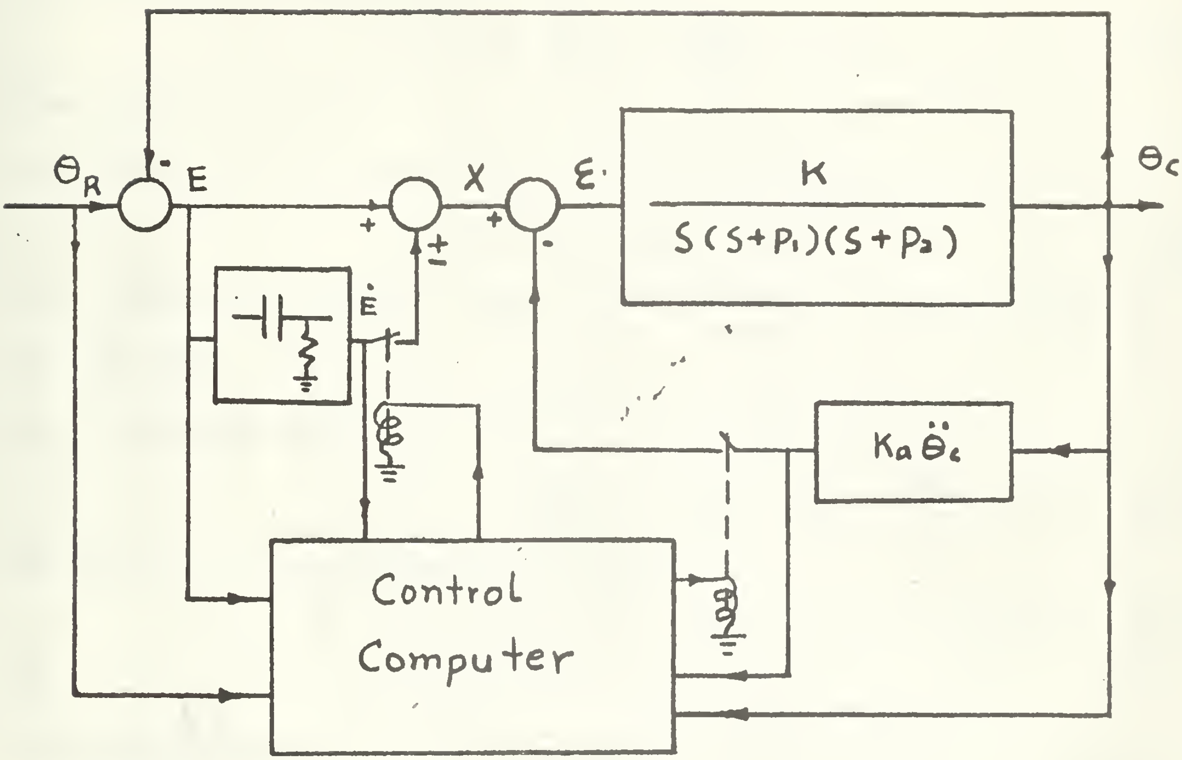


Fig. 7-12 Block Diagram of a 3rd Order Type One System with Discontinuous Feedback and Feedforward Compensation.

SECTION VIII - TRANSLATED HYPERPLANE AND DIVIDED PHASE SPACE THEORY APPLIED TO 3RD ORDER TYPE TWO SYSTEMS WITH DISCONTINUOUS FEEDFORWARD COMPENSATIONS.

From Fig. 7-13

$$\dddot{\theta}_c + a\ddot{\theta}_c = K\mathcal{E} = K(E + \ell_T\dot{E} + \ell_a\ddot{E}) \tag{7-25}$$

$$\dddot{E} + (a + K\ell_{Ra})\ddot{E} + K\ell_{RT}\dot{E} + KE = \dddot{\theta}_R + a\,\ddot{\theta}_R \tag{7-26}$$

For step input and ramp input, $\dddot{\theta}_R = \ddot{\theta}_R = 0$

$$\dddot{E} + (a + K\ell_{Ra})\ddot{E} + K\ell_{RT}\dot{E} + KE = 0 \tag{7-27}$$

Assume the real roots are r_1, r_2, and r_3, and r_1, is very small.

From Chapter II, the equation of the fast hyperplane is

$$U = \ddot{E} + (r_2 + r_3)\dot{E} + r_2 r_3 E = 0 \tag{7-28}$$

This is one of the boundary planes of the divided error space for step and ramp input.

For acceleration input $\theta_R = \alpha_i t^2$, $\dddot{\theta}_R = 0$, $\ddot{\theta}_R = 2\alpha_i$

$$\dddot{E} + (a + K\ell_{Ra})\ddot{E} + K\ell_{RT}\dot{E} + K(E - \frac{2\,a\alpha_i}{K}) = 0 \tag{7-29}$$

The equation of the translated hyperplane is

$$U = \ddot{E} + (r_2 + r_3)\dot{E} + r_2 r_3 \left(E - \frac{2a\alpha_i}{K}\right) = 0 \tag{7-30}$$

For positive acceleration input, this equation corresponds to a plane parallel to the original hyperplane for step input. The point of intersection one the E axis is $\frac{2a\alpha_i}{K}$. For optimum acceleration input response the point of intersection of the $\ddot{E}$ axis has to be α_i then

$$\gamma_2\gamma_3\,\frac{2a\alpha_i}{K} = \alpha_i$$

i.e.
$$\gamma_2\gamma_3 = \frac{K}{2a} \tag{7-31}$$

If divided error space theory is not used the system will have acceleration overshoot or undershoot unless the product of γ_2 and γ_3 is chosen according to the above relation. The value of γ_2, γ_3 has to be equal to or larger than $\frac{K}{2a}$ even in the divided error space, just like in the 2nd order case.

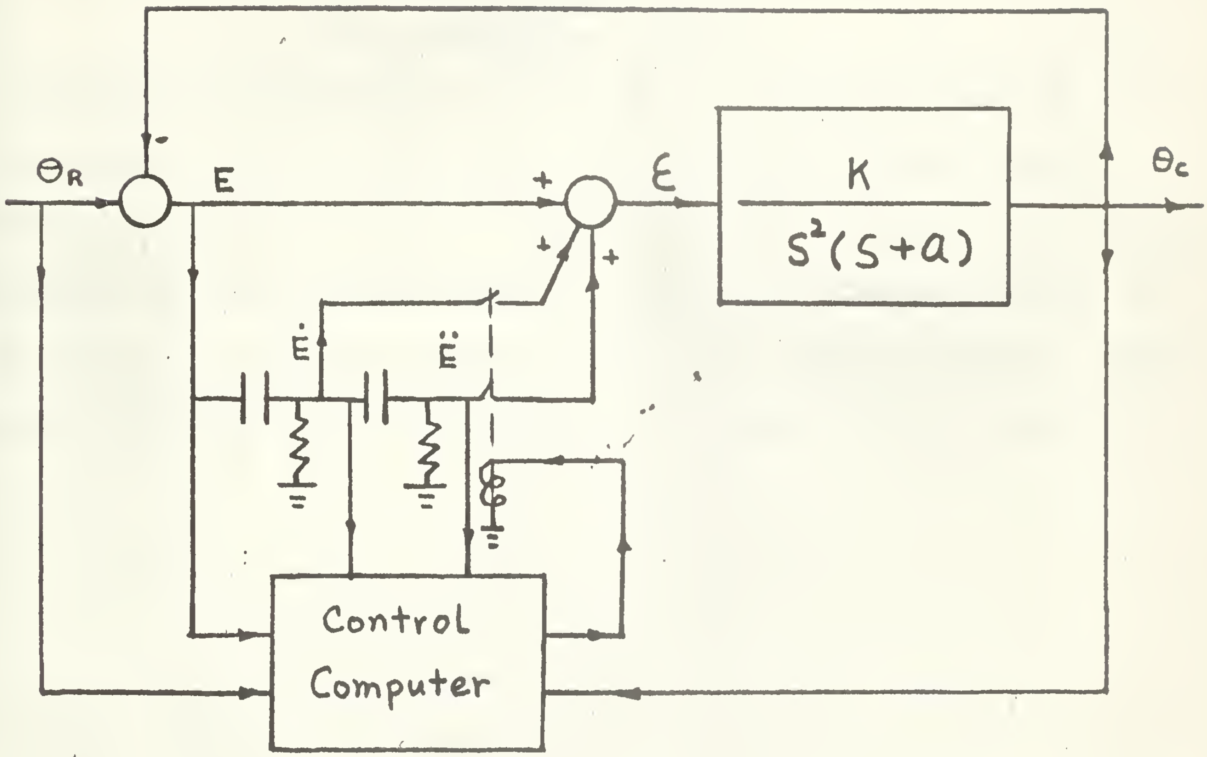


Fig. 7-13 Block Diagram of a 3rd Order Type Two System with Feedforward Compensation.

The starting point of the positive ramp input is desired to be under the translated fast eigenvector. A sketch of this hyperplane is in Fig. 7-14.

Moreover if the point of intersection of this translated hyperplane on the $\dot{E}$ axis is specified to be equal to or larger than a definite value ωi then another relation between γ_2 and γ_3 is

$$(\gamma_2 + \gamma_3)\dot{E} + \gamma_2\gamma_3\left(-\frac{2a\alpha i}{K}\right) = 0$$

or

$$\dot{E} = \frac{\gamma_2\gamma_3 \frac{2a\alpha i}{K}}{(\gamma_2 + \gamma_3)} \geq \omega i \qquad (7\text{-}32)$$

Therefore both the values of γ_2 and γ_3 can be specified and the gains of feedforward paths can be calculated. This does not mean that only one particular set of roots can give optimum response for step, ramp and acceleration inputs, the relations just derived are guides for choosing the values of real roots. In case heavily damped compensation cannot be obtained, these relations will indicate how the trajectories are shaped near the hyperplane.

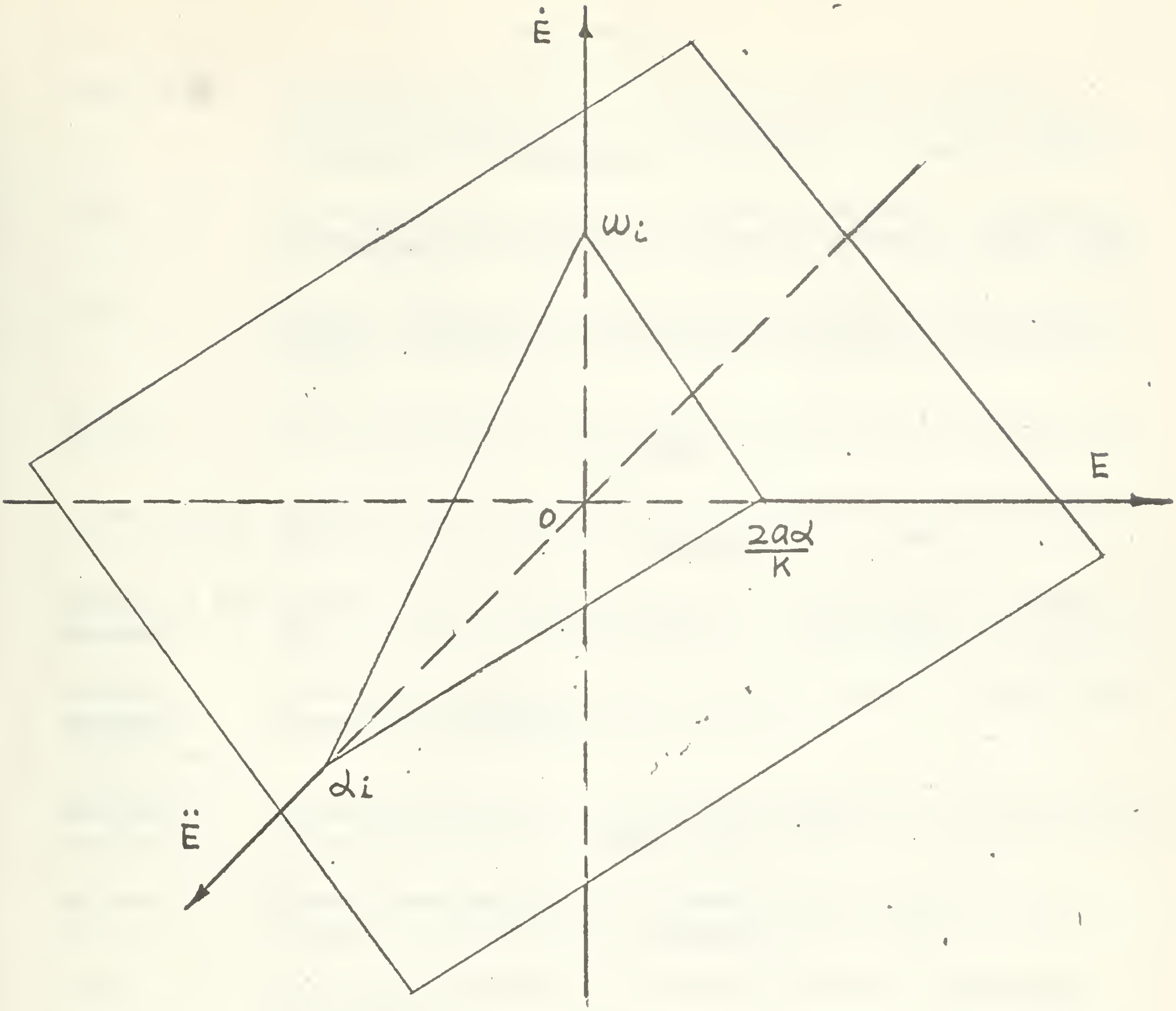


Fig. 7-14 A sketch to Show the Intersection Points of the Hyperplane on the Axes.

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